Solutions to Review Problems for Exam 2

1. A random point (X, Y) is distributed uniformly on the square with vertices $(-1, -1)$, $(1, -1)$, $(1, 1)$ and $(-1, 1)$.

   (a) Give the joint pdf for X and Y.

   (b) Compute the following probabilities:

      (i) $\Pr(X^2 + Y^2 < 1)$,

      (ii) $\Pr(2X - Y > 0)$,

      (iii) $\Pr(|X + Y| < 2)$.

   ***Solution**:* The square is pictured in Figure 1 and has area 4.

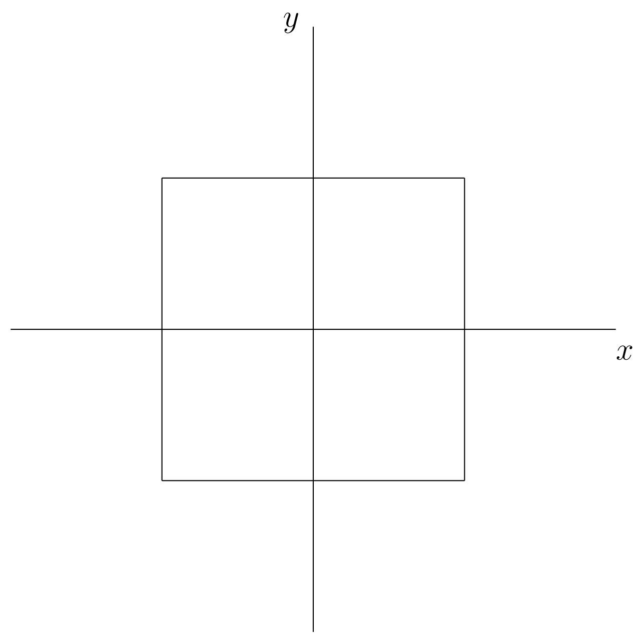

Figure 1: Sketch of square in Problem 1

   (a) Consequently, the joint pdf of (X, Y) is given by

$$f_{(X,Y)}(x, y) = \begin{cases} \dfrac{1}{4}, & \text{for } -1 < x < 1, -1 < y < 1; \\ \\ 0 & \text{elsewhere.} \end{cases} \tag{1}$$

(b) Denoting the square in Figure 1 by R, it follows from (1) that, for any subset A of $\mathbb{R}^2$,

$$\Pr[(x,y) \in A] = \iint_A f_{(X,Y)}(x,y)\; dxdy = \frac{1}{4} \cdot \text{area}(A \cap R); \quad (2)$$

that is, $\Pr[(x,y) \in A]$ is one–fourth the area of the portion of A in R.

We will use the formula in (2) to compute each of the probabilities in (i), (ii) and (iii).

(i) In this case, A is the circle of radius 1 around the origin in $\mathbb{R}^2$ and pictured in Figure 2.

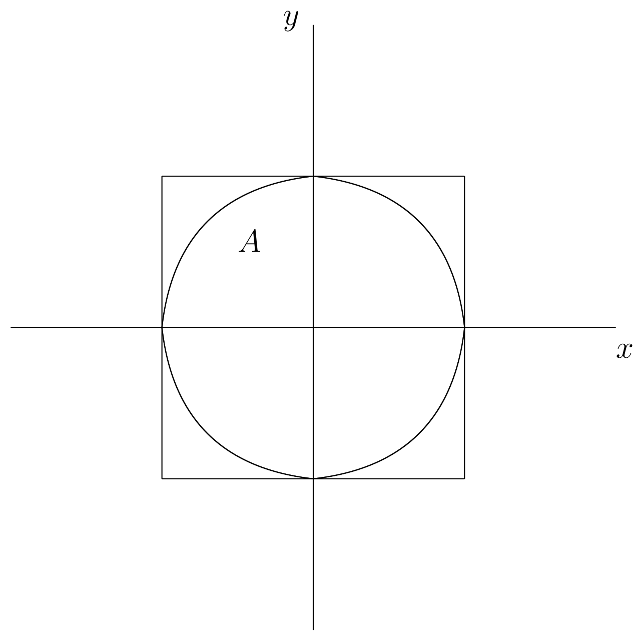

Figure 2: Sketch of A in Problem 1(i)

Note that the circle A in Figure 2 is entirely contained in the square R so that, by the formula in (2),

$$\Pr(X^2 + Y^2 < 1) = \frac{\text{area}(A)}{4} = \frac{\pi}{4}.$$

(ii) The set A in this case is pictured in Figure 3 on page 3. Thus, in this case, $A \cap R$ is a trapezoid of area $2 \cdot \dfrac{\frac{1}{2} + \frac{3}{2}}{2} = 2,$ so

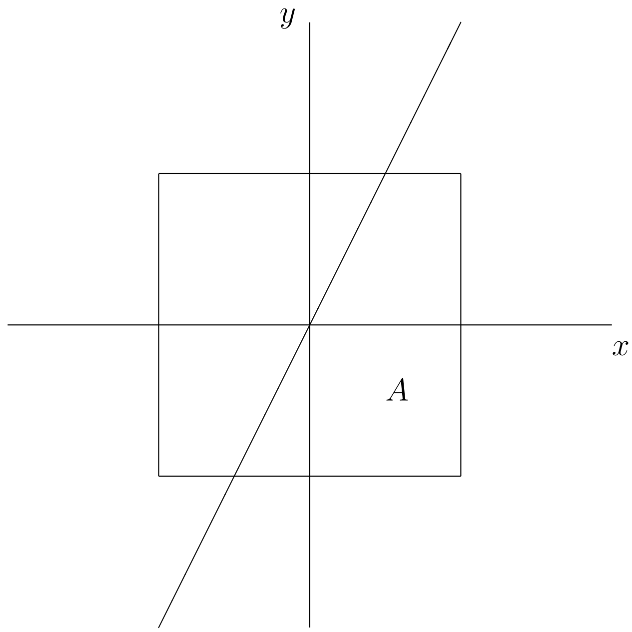

Figure 3: Sketch of A in Problem 1(ii)

that, by the formula in (2),

$$\Pr(2X - Y > 0) = \frac{1}{4} \cdot \text{area}(A \cap R) = \frac{1}{2}.$$

(iii) In this case, A is the region in the xy–plane between the lines $x + y = 2$ and $x + y = -2$ (see Figure 4 on page 4). Thus, $A \cap R$ is R so that, by the formula in (2),

$$\Pr(|X + Y| < 2) = \frac{\text{area}(R)}{4} = 1.$$

□

2. Let $F_{(X,Y)}$ be the joint cdf of two random variables X and Y. For real constants $a < b$, $c < d$, show that

$$\Pr(a < X \leqslant b, c < Y \leqslant d) = F_{(X,Y)}(b,d) - F_{(X,Y)}(b,c) - F_{(X,Y)}(a,d) + F_{(X,Y)}(a,c).$$

Use this result to show that $F(x,y) = \begin{cases} 1 & \text{if } x + 2y \geqslant 1, \\ 0 & \text{otherwise,} \end{cases}$ cannot be the joint cdf of two random variables.

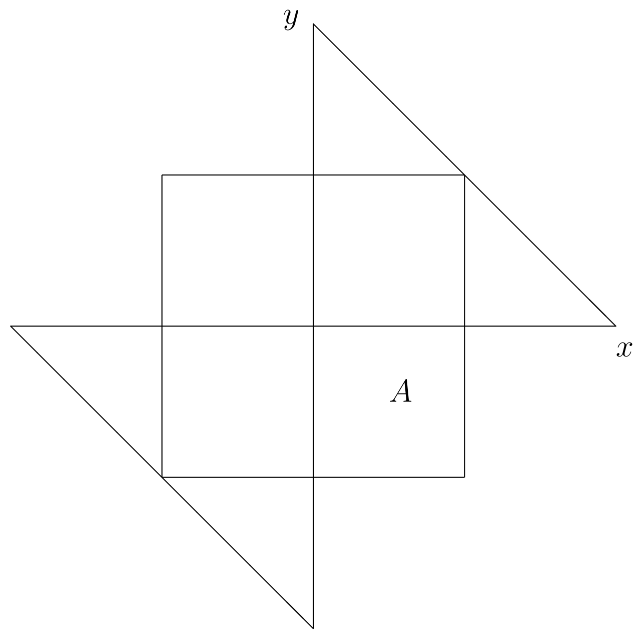

Figure 4: Sketch of A in Problem 1(iii)

***Solution**:* Let $A = \{(x, y) \in \mathbb{R}^2 \mid a < x \leqslant b,\ c < y \leqslant d\}$; we then want to compute $\Pr[(X, Y) \in A]$.
We also define the following events:

$$A_1 = \{(x, y) \in \mathbb{R}^2 \mid x \leqslant b,\ y \leqslant d\},$$

$$A_2 = \{(x, y) \in \mathbb{R}^2 \mid x \leqslant a,\ y \leqslant c\},$$

$$A_3 = \{(x, y) \in \mathbb{R}^2 \mid x \leqslant a,\ c < y \leqslant d\},$$

and

$$A_4 = \{(x, y) \in \mathbb{R}^2 \mid a < x \leqslant b,\ y \leqslant c\}.$$

Then, A_1 is a disjoint union of the events A, A_2, A_3 and A_4 (see Figure 5). It then follows that

$$\Pr[(X, Y) \in A_1] = \Pr[(X, Y) \in A \cup A_2 \cup A_3 \cup A_4]$$

or

$$\begin{aligned} \Pr[(X, Y) \in A_1] &= \Pr[(X, Y) \in A] + \Pr[(X, Y) \in A_2] + \\ &\quad + \Pr[(X, Y) \in A_3] + \Pr[(X, Y) \in A_4]. \end{aligned} \qquad (3)$$

Observe that

$$\Pr[(X, Y) \in A_1] = \Pr(X \leqslant b, Y \leqslant d) = F_{(X,Y)}(b, d)$$

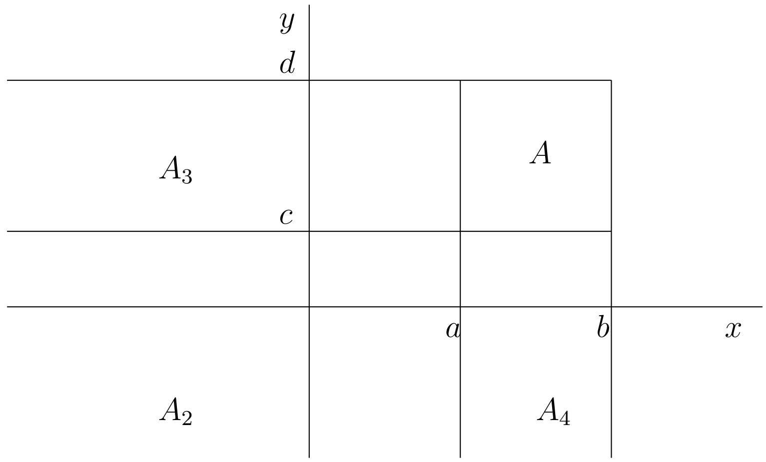

Figure 5: Events A, A_1, A_2, A_3 and A_4 in the xy–plane

and

$$\Pr[(X,Y) \in A_2] = \Pr(X \leqslant a, Y \leqslant c) = F_{(X,Y)}(a,c).$$

It then follows from equation (3) that

$$\begin{aligned} \Pr[(X,Y) \in A] &= F_{(X,Y)}(b,d) - F_{(X,Y)}(a,c) \\ &\quad -\Pr[(X,Y) \in A_3] - \Pr[(X,Y) \in A_4]. \end{aligned} \tag{4}$$

On the other hand, observe that

$$\Pr[(X,Y) \in A_3 \cup A_2] = \Pr(X \leqslant a, Y \leqslant d) = F_{(X,Y)}(a,d) \tag{5}$$

and

$$\Pr[(X,Y) \in A_4 \cup A_2] = \Pr(X \leqslant b, Y \leqslant c) = F_{(X,Y)}(b,c). \tag{6}$$

Moreover,

$$\begin{aligned} \Pr[(X,Y) \in (A_3 \cup A_2) \cup (A_4 \cup A_2)] &= \Pr[(X,Y) \in A_3 \cup A_2] \\ &\quad +\Pr[(X,Y) \in A_4 \cup A_2] \\ &\quad -\Pr[(X,Y) \in A_2], \end{aligned}$$

since $(A_3 \cup A_2) \cap (A_4 \cup A_2) = A_2$. It then follows from equations (5) and (6) that

$$\begin{aligned} \Pr[(X,Y) \in (A_3 \cup A_2) \cup (A_4 \cup A_2)] &= F_{(X,Y)}(a,d) \\ &\quad +F_{(X,Y)}(b,c) \\ &\quad -F_{(X,Y)}(a,c). \end{aligned}$$

However, since $(A_3 \cup A_2) \cup (A_4 \cup A_2) = A_2 \cup A_3 \cup A_4$, we also get that

$$\begin{aligned} \Pr[(X,Y) \in (A_3 \cup A_2) \cup (A_4 \cup A_2)] &= \Pr[(X,Y) \in A_2] \\ &\quad + \Pr[(X,Y) \in A_3] \\ &\quad + \Pr[(X,y) \in A_4]. \end{aligned}$$

We therefore get, using $\Pr[(X,Y) \in A_2] = F_{(X,Y)}(a,c)$, that

$$\begin{aligned} \Pr[(X,Y) \in A_3] + \Pr[(X,Y) \in A_4] &= F_{(X,Y)}(a,d) + F_{(X,Y)}(b,c) \\ &\quad -2F_{(X,Y)}(a,c). \end{aligned}$$

Substituting this into equation (4) yields

$$\begin{aligned} \Pr[(X,Y) \in A] &= F_{(X,Y)}(b,d) - F_{(X,Y)}(a,c) \\ &\quad -F_{(X,Y)}(a,d) - F_{(X,Y)}(b,c) + 2F_{(X,Y)}(a,c), \end{aligned}$$

from which we get

$$\Pr[(X,Y) \in A] = F_{(X,Y)}(b,d) - F_{(X,Y)}(a,d) - F_{(X,Y)}(b,c) + F_{(X,Y)}(a,c).$$

Next, suppose that $F(x,y) = \begin{cases} 1 & \text{if } x + 2y \geqslant 1, \\ 0 & \text{otherwise,} \end{cases}$ is the joint cdf of two random variables X and Y. Consider the set

$$A = \{(x,y) \in \mathbb{R}^2 \mid 0 < x \leqslant 1, 0 < y \leqslant 1/2\}.$$

By what we just proved,

$$\begin{aligned} \Pr[(X,Y) \in A] &= F(1,1/2) - F(0,1/2) - F(1,0) + F(0,0) \\ &= 1 - 1 - 1 + 0 \\ &= -1 < 0, \end{aligned}$$

which is impossible since $\Pr[(X,Y) \in A] \geqslant 0$. Therefore, F cannot be a joint pdf. □

3. The random pair (X,Y) has the joint distribution shown in Table 1.

   (a) Show that X and Y are not independent.

$X\backslash Y$	2	3	4
1	$\frac{1}{12}$	$\frac{1}{6}$	0
2	$\frac{1}{6}$	0	$\frac{1}{3}$
3	$\frac{1}{12}$	$\frac{1}{6}$	0

Table 1: Joint Probability Distribution for X and Y, $p_{(X,Y)}$

$X\backslash Y$	2	3	4	p_X
1	$\frac{1}{12}$	$\frac{1}{6}$	0	$\frac{1}{4}$
2	$\frac{1}{6}$	0	$\frac{1}{3}$	$\frac{1}{2}$
3	$\frac{1}{12}$	$\frac{1}{6}$	0	$\frac{1}{4}$
p_Y	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

Table 2: Joint pdf for X and Y and marginal distributions p_X and p_Y

Solution: Table 2 shows the marginal distributions of X and Y on the margins.
Observe from Table 2 that

$$p_{(X,Y)}(1,4) = 0,$$

while

$$p_X(1) = \frac{1}{4} \quad \text{and} \quad p_Y(4) = \frac{1}{3}.$$

Thus,

$$p_X(1) \cdot p_Y(4) = \frac{1}{12};$$

so that

$$p_{(X,Y)}(1,4) \neq p_X(1) \cdot p_Y(4),$$

and, therefore, X and Y are not independent. □

(b) Give a probability table for random variables U and V that have the same marginal distributions as X and Y, respectively, but are independent.

Solution: Table 3 on page 8 shows the joint pmf of (U,V) and the marginal distributions, p_U and p_V. □

4. Let X denote the number of trials needed to obtain the first head, and let Y be the number of trials needed to get two heads in repeated tosses of a fair coin. Are X and Y independent random variables?

$U\backslash V$	2	3	4	p_U
1	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$
2	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$
3	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$
p_V	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

Table 3: Joint pdf for U and V and their marginal distributions.

Solution: X has a geometric distribution with parameter $p = \frac{1}{2}$, so that

$$p_X(k) = \frac{1}{2^k}, \quad \text{for } k = 1, 2, 3, \ldots \tag{7}$$

On the other hand,

$$\Pr[Y = 2] = \frac{1}{4}, \tag{8}$$

since, in two repeated tosses of a coin, the events are HH, HT, TH and TT, and these events are equally likely.
Next, consider the joint event $(X = 2, Y = 2)$. Note that

$$(X = 2, Y = 2) = [X = 2] \cap [Y = 2] = \emptyset,$$

since $[X = 2]$ corresponds to the event TH, while $[Y = 2]$ to the event HH. Thus,

$$\Pr(X = 2, Y = 2) = 0,$$

while

$$p_X(2) \cdot p_Y(2) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16},$$

by (7) and (8). Thus,

$$p_{(X,Y)}(2, 2) \neq p_X(2) \cdot p_X(2).$$

Hence, X and Y are not independent. □

5. Prove that if the joint cdf of X and Y satisfies

$$F_{(X,Y)}(x, y) = F_X(x)F_Y(y), \tag{9}$$

then for any pair of intervals (a, b) and (c, d),

$$\Pr(a < X \leq b, c < Y \leq d) = \Pr(a < X \leqslant b)\Pr(c < Y \leqslant d).$$

Solution: We use the result of Problem 3 in this review sheet:

$$\Pr(a < X \leq b, c < Y \leq d) = F_{(X,Y)}(b,d) - F_{(X,Y)}(b,c) - F_{(X,Y)}(a,d) + F_{(X,Y)}(a,c);$$

thus, using the assumption in (9),

$$\begin{aligned}
\Pr(a < X \leq b, c < Y \leq d) &= F_X(b)F_Y(d) - F_X(b)F_Y(c) \\
&\quad - F_X(a)F_Y(d) + F_X(a)F_Y(c) \\
&= (F_X(b) - F_X(a))F_Y(d) \\
&\quad - (F_X(b) - F_X(a))F_Y(c) \\
&= (F_X(b) - F_X(a))(F_Y(d) - F_Y(c)) \\
&= \Pr(a < X \leqslant b)\Pr(c < Y \leqslant d),
\end{aligned}$$

which was to be shown. □

6. Let $g(t)$ denote a non–negative, integrable function of a single variable with the property that

$$\int_0^\infty g(t) \; dt = 1.$$

Define

$$f(x,y) = \begin{cases} \dfrac{2g(\sqrt{x^2+y^2})}{\pi\sqrt{x^2+y^2}} & \text{for } 0 < x < \infty, \; 0 < y < \infty, \\ \\ 0\text{otherwise.} \end{cases}$$

Show that $f(x,y)$ is a joint pdf for two random variables X and Y.

Solution: First observe that f is non–negative since g is non–negative. Next, compute

$$\iint_{\mathbb{R}^2} f(x,y) \; dx \; dy \;\; = \;\; \int_0^\infty \int_0^\infty \frac{2g(\sqrt{x^2+y^2})}{\pi\sqrt{x^2+y^2}} \; dx \; dy.$$

Switching to polar coordinates we then get that

$$\begin{aligned}\iint_{\mathbb{R}^2} f(x,y)\ dx\ dy &= \int_0^{\pi/2}\int_0^\infty \frac{2g(r)}{\pi r}\ r\ dr\ d\theta \\ &= \frac{\pi}{2}\int_0^\infty \frac{2}{\pi} g(r)\ dr \\ &= \int_0^\infty g(r)\ dr \\ &= 1,\end{aligned}$$

and therefore $f(x,y)$ is indeed a joint pdf for two random variables X and Y. □

7. Let $X \sim \text{Exponential}(1)$, and define Y to be the integer part of $X+1$; that is, $Y = i+1$ if and only if $i \leqslant X < i+1$, for $i = 0, 1, 2, \ldots$ Find the pmf of Y, and deduce that $Y \sim \text{Geometric}(p)$ for some $0 < p < 1$. What is the value of p?

***Solution**:* Compute

$$\Pr[Y = i+1] = \Pr[i \leqslant X < i+1] = \Pr[i < X \leqslant i+1],$$

since X is continuous; so that

$$\Pr[Y = i+1] = \int_i^{i+1} f_X(x)\ dx, \tag{10}$$

where

$$f_X(x) = \begin{cases} e^{-x} & \text{if } x > 0; \\ 0 & \text{if } x \leqslant 0, \end{cases} \tag{11}$$

given that $X \sim \text{Exponential}(1)$.

Evaluating the integral in (10), for $i \geqslant 0$ and f_X as given in (11), yields

$$\begin{aligned}\Pr[Y = i+1] &= \int_i^{i+1} e^{-x}\ dx \\ &= \left[-e^{-x}\right]_i^{i+1} \\ &= e^{-i} - e^{-i-1},\end{aligned}$$

so that

$$\Pr[Y = i + 1] \;=\; \left(\frac{1}{e}\right)^i \left(1 - \frac{1}{e}\right) \qquad (12)$$

It follows from (12) that $Y \sim \text{Geometric}(p)$ with $p = 1 - \dfrac{1}{e}$. □

8. Suppose that two persons make an appointment to meet between 5 PM and 6 PM at a certain location and they agree that neither person will wait more than 10 minutes for each person. If they arrive independently at random times between 5 PM and 6 PM, what is the probability that they will meet?

   ***Solution**:* Let X denote the arrival time of the first person and Y that of the second person. Then X and Y are independent and uniformly distributed on the interval (5 PM, 6 PM), in hours. It then follows that the joint pdf of X and Y is

$$f_{(X,Y)}(x, y) = \begin{cases} 1, & \text{if } 5 \text{ PM} < x < 6 \text{ PM}, 5 \text{ PM} < x < 6 \text{ PM}, \\ 0, & \text{elsewhere.} \end{cases}$$

   Define $W = |X - Y|$; this is the time that one person would have to wait for the other one. Then, W takes on values, w, between 0 and 1 (in hours). The probability that that a person would have to wait more than 10 minutes is

$$\Pr(W > 1/6),$$

   since the time is being measured in hours. It then follows that the probability that the two persons will meet is

$$1 - \Pr(W > 1/6) = \Pr(W \leqslant 1/6) = F_W(1/6).$$

   We will therefore first find the cdf of W. To do this, we compute

$$\begin{aligned} \Pr(W \leqslant w) &= \Pr(|X - Y| \leqslant w), \quad \text{for } 0 < w < 1, \\ &= \iint_A f_{(X,Y)}(x, y) \; dx \; dy, \end{aligned}$$

   where A is the event

$$A = \{(x, y) \in \mathbb{R}^2 \mid 5 \text{ PM} < x < 6 \text{ PM}, 5 \text{ PM} < y < 6 \text{ PM}, |x - y| \leqslant w\}.$$

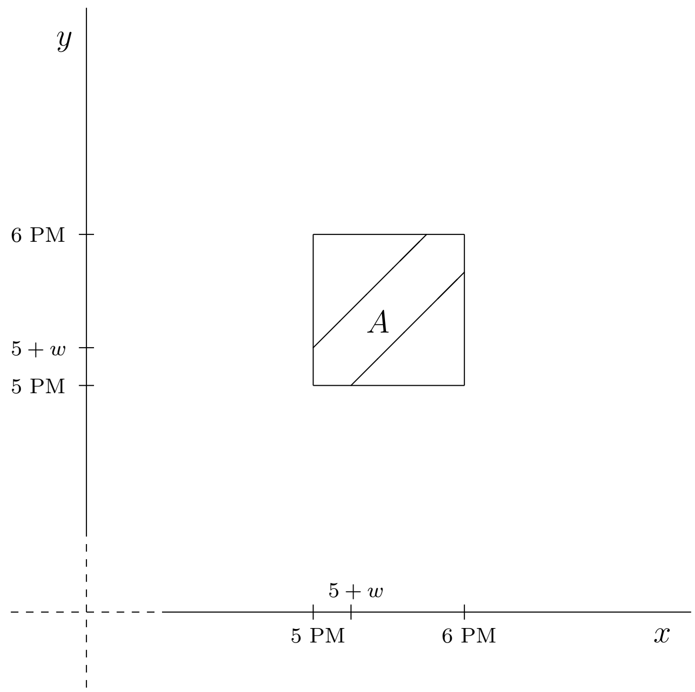

Figure 6: Event A in the xy–plane

This event is pictured in Figure 6.
We then have that

$$
\begin{aligned}
\Pr(W \leqslant w) &= \iint_A \, dx \, dy \\
&= \text{area}(A),
\end{aligned}
$$

where the area of A can be computed by subtracting from 1 the area of the two corder triangles shown in Figure 6:

$$
\begin{aligned}
\Pr(W \leqslant w) &= 1 - (1-w)^2 \\
&= 2w - w^2.
\end{aligned}
$$

Consequently, $F_W(w) = 2w - w^2$ for $0 < w < 1$. Thus the probability that the two persons will meet is

$$
F_W(1/6) = 2 \cdot \frac{1}{6} - \left(\frac{1}{6}\right)^2 = \frac{11}{36},
$$

or about 30.56%. □

9. Suppose that a book with n pages contains on average λ misprints per page. What is the probability that there will be at least m pages which contain more than k missprints?

   Solution: Let Y denote the number of misprints in one page. Then, we may assume that Y follows a Poisson(λ) distribution; so that

   $$\Pr[Y = r] = \frac{\lambda^r}{r!}\, e^{-\lambda}, \quad \text{for } r = 0, 1, 2, \ldots$$

   Thus, the probability that there will be more than k missprints in a given page is

   $$\begin{aligned} p &= \Pr(Y > k) \\ &= 1 - \Pr(Y \leqslant k), \end{aligned}$$

   so that

   $$p = 1 - \sum_{r=0}^{k} \frac{\lambda^r}{r!}\, e^{-\lambda}. \tag{13}$$

   Next, let X denote the number of the pages out of the n that contain more than k missprints. Then, $X \sim$ Binomial(n, p), where p is as given in (13). Then the probability that there will be at least m pages which contain more than k missprints is

   $$\Pr[X \geqslant m] = \sum_{\ell=m}^{n} \binom{n}{\ell} p^\ell (1-p)^{n-\ell},$$

   where

   $$p = 1 - \sum_{r=0}^{k} \frac{\lambda^r}{r!}\, e^{-\lambda}.$$

   □

10. Suppose that the total number of items produced by a certain machine has a Poisson distribution with mean λ, all items are produced independently of one another, and the probability that any given item produced by the machine will be defective is p.

    Let X denote the number of defective items produced by the machine.

    (a) Determine the marginal distribution of the number of defective items, X.

Solution: Let N denote the number of items produced by the machine. Then,

$$N \sim \text{Poisson}(\lambda), \tag{14}$$

so that

$$\Pr[N = n] = \frac{\lambda^n}{n!}\, e^{-\lambda}, \qquad \text{for } n = 0, 1, 2, \ldots$$

Now, since all items are produced independently of one another, and the probability that any given item produced by the machine will be defective is p, X has a conditional distribution (conditioned on $N = n$) that is Binomial(n, p); thus,

$$\Pr[X = k \mid N = n] = \begin{cases} \dbinom{n}{k} p^k (1-p)^{n-k}, & \text{for } k = 0, 1, 2, \ldots, n; \\ \\ 0 & \text{elsewhere.} \end{cases} \tag{15}$$

Then,

$$\begin{aligned} \Pr[X = k] &= \sum_{n=0}^{\infty} \Pr[X = k, N = n] \\ &= \sum_{n=0}^{\infty} \Pr[N = n] \cdot \Pr[X = k \mid N = n], \end{aligned}$$

where $\Pr[X = k \mid N = n] = 0$ for $n < k$, so that, using (14) and (15),

$$\begin{aligned} \Pr[X = k] &= \sum_{n=k}^{\infty} \frac{\lambda^n}{n!}\, e^{-\lambda} \cdot \binom{n}{k} p^k (1-p)^{n-k} \\ &= \frac{e^{-\lambda}}{k!}\, p^k \sum_{n=k}^{\infty} \lambda^n \frac{1}{(n-k)!} (1-p)^{n-k}. \end{aligned} \tag{16}$$

Next, make the change of variables $\ell = n - k$ in the last summation in (16) to get

$$\Pr[X = k] = \frac{e^{-\lambda}}{k!}\, p^k \sum_{\ell=0}^{\infty} \lambda^{\ell+k} \frac{1}{\ell!} (1-p)^{\ell},$$

so that

$$\begin{aligned}\Pr[X = k] &= \frac{(\lambda p)^k}{k!}\, e^{-\lambda} \sum_{\ell=0}^{\infty} \frac{1}{\ell!}[\lambda(1-p)]^{\ell} \\ &= \frac{(\lambda p)^k}{k!}\, e^{-\lambda} e^{\lambda(1-p)} \\ &= \frac{(\lambda p)^k}{k!}\, e^{-\lambda p},\end{aligned}$$

which shows that

$$X \sim \text{Poisson}(\lambda p). \tag{17}$$

□

(b) Let Y denote the number of non–defective items produced by the machine. Show that X and Y are independent random variables.

Solution: Similar calculations to those leading to (17) show that

$$Y \sim \text{Poisson}(\lambda(1-p)), \tag{18}$$

since the probability of an item coming out non–defective is $1-p$. Next, observe that $Y = N - X$ and compute the joint probability

$$\begin{aligned}\Pr[X = k, Y = \ell] &= \Pr[X = k, N = k+\ell] \\ &= \Pr[N = k+\ell] \cdot \Pr[X = k \mid N = k+\ell] \\ &= \frac{\lambda^{k+\ell}}{(k+\ell)!}\, e^{-\lambda} \cdot \binom{k+\ell}{k} p^k (1-p)^{\ell}\end{aligned}$$

by virtue of (14) and (15). Thus,

$$\begin{aligned}\Pr[X = k, Y = \ell] &= \frac{\lambda^{k+\ell}}{k!\ \ell!}\, e^{-\lambda}\ p^k (1-p)^{\ell} \\ &= \frac{\lambda^{k+\ell}}{k!\ \ell!}\, e^{-\lambda}\ p^k (1-p)^{\ell},\end{aligned}$$

where

$$e^{-\lambda} = e^{-[p+(1-p)]\lambda} = e^{-p\lambda} \cdot e^{-(1-p)\lambda}.$$

Thus,

$$\begin{aligned}\Pr[X = k, Y = \ell] &= \frac{(p\lambda)^k}{k!}\, e^{-p\lambda} \cdot \frac{[(1-p)\lambda]^\ell}{\ell!}\, e^{-(1-p)\lambda} \\ &= p_{_X}(k) \cdot p_{_Y}(\ell),\end{aligned}$$

in view of (17) and (18). Hence, X and Y are independent. □

11. Suppose that the proportion of color blind people in a certain population is 0.005. Estimate the probability that there will be more than one color blind person in a random sample of 600 people from that population.

    ***Solution**:* Set $p = 0.005$ and $n = 600$. Denote by Y the number of color blind people in the sample. Then, we may assume that $Y \sim \text{Binomial}(n, p)$. Since p is small and n is large, we may use the Poisson approximation to the binomial distribution to get

    $$\Pr[Y = k] \approx \frac{\lambda^k}{k!}\, e^{-\lambda},$$

    where $\lambda = np = 3$.
    Then,

    $$\begin{aligned}\Pr[Y > 1] &= 1 - \Pr[Y \leqslant 1] \\ &\approx 1 - e^{-3} - 3e^{-3} \\ &\approx 0.800852.\end{aligned}$$

    Thus, the probability that there will be more than one color blind person in a random sample of 600 people from that population is about 80%. □

12. An airline sells 200 tickets for a certain flight on an airplane that has 198 seats because, on average, 1% of purchasers of airline tickets do not appear for departure of their flight. Estimate the probability that everyone who appears for the departure of this flight will have a seat.

***Solution**:* Set $p = 0.01$, $n = 200$ and let Y denote the number of ticket purchasers that do not appear for departure. Then, we may assume that $Y \sim \text{Binomial}(n, p)$. We want to estimate the probability $\Pr[Y \geqslant 2]$. Using the Poisson(λ), with $\lambda = np = 2$, approximation to the distribution of Y, we get

$$\begin{aligned} \Pr[Y \geqslant 2] &= 1 - \Pr[Y \leqslant 1] \\ &\approx 1 - e^{-2} - 2e^{-2} \\ &\approx 0.5940. \end{aligned}$$

Thus, the probability that everyone who appears for the departure in this flight will have a seat is about 59.4%. □

13. Let X and Y denote random variables. Show that

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y).$$

Deduce that, if X and Y are independent, then

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y).$$

***Solution**:* We compute

$$\text{Var}(X + Y) = E[[X + Y - (\mu_X + \mu_Y)]^2],$$

where $\mu_X = E(X)$, $\mu_Y = E(Y)$, and

$$\begin{aligned} [X + Y - (\mu_X + \mu_Y)]^2 &= [(X - \mu_X) + (Y - \mu_Y)]^2 \\ &= (X - \mu_X)^2 + 2(X - \mu_X)(Y - \mu_Y) + (Y - \mu_Y)^2; \end{aligned}$$

so that, using the linearity of the expectation and the definition of covariance

$$\begin{aligned} \text{Var}(X + Y) &= E[(X - \mu_X)^2] + 2E[(X - \mu_X)(Y - \mu_Y)] + E[(Y - \mu_Y)^2] \\ &= \text{Var}(X) + 2\text{Cov}(X, Y) + \text{Var}(Y), \end{aligned}$$

which was to be shown.

Now, if X and Y are independent, then $\text{Cov}(X, Y) = 0$. We therefore get that, if X and Y are independent, then

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y).$$

□