Solutions to Exam 3 (Part II)

1. A company manufactures a brand of incandescent light bulbs. Assume that the light bulbs have a lifetime in months that is normally distributed with mean 3.5 and variance 1; assume also that the lifetimes of the light bulbs are independent. A consumer buys a number of these bulbs with the intention of replacing them successively as they burn out. What is the smallest number of bulbs to be purchased so that the succession of light bulbs produces light for at least 47 months with probability at least 97.5%? Explain your reasoning.

   Solution: Let $X_1, X_2, \ldots, X_n$ denote the lifetimes of the n light bulbs in months. We are given that $X_k \sim \text{Normal}(3.5, 1)$ and that $X_1, X_2, \ldots, X_n$ are independent random variables. It then follows that

$$\sum_{k=1}^{n} X_k \sim \text{Normal}(3.5n, n)$$

and, therefore,

$$\frac{\displaystyle\sum_{k=1}^{n} X_k - 3.5n}{\sqrt{n}} \sim \text{Normal}(0, 1). \tag{1}$$

We would like to find n so that

$$\Pr\left(\sum_{k=1}^{n} X_k > 47\right) \geqslant 0.975,$$

or

$$\Pr\left(\frac{\displaystyle\sum_{k=1}^{n} X_k - 3.5n}{\sqrt{n}} > \frac{47 - 3.5n}{\sqrt{n}}\right) \geqslant 0.975;$$

so that, in view of (1)

$$1 - F_{Z}\left(\frac{47 - 3.5n}{\sqrt{n}}\right) \geqslant 0.975,$$

where $Z \sim \text{Normal}(0, 1)$; or

$$F_{Z}\left(\frac{47 - 3.5n}{\sqrt{n}}\right) \leqslant 0.025, \tag{2}$$

where $Z \sim \text{Normal}(0,1)$.

By virtue of the symmetry of the pdf of the standard normal distribution, (2) can be attained if we set

$$\frac{47 - 3.5n}{\sqrt{n}} \leqslant -z^*, \tag{3}$$

where $z^* > 0$ is such that $F_Z(z^*) = 0.975$; that is, $z^* \doteq 1.96$, which we can round up to $z^* \doteq 2$. We therefore obtain from (3) that

$$\frac{47 - 3.5n}{\sqrt{n}} \leqslant -2,$$

or

$$3.5n - 47 \geqslant 2\sqrt{n}. \tag{4}$$

We can solve the inequality in (4) by first writing

$$n - \frac{4}{7}\sqrt{n} \geqslant \frac{94}{7}$$

and then completing the square,

$$n - \frac{4}{7}\sqrt{n} + \frac{4}{49} \geqslant \frac{94}{7} + \frac{4}{49},$$

to obtain

$$\left(\sqrt{n} - \frac{2}{7}\right)^2 \geqslant \frac{662}{49},$$

or

$$\sqrt{n} - \frac{2}{7} \geqslant \frac{\sqrt{662}}{7};$$

so that

$$\sqrt{n} \geqslant \frac{2 + \sqrt{662}}{7},$$

from which we get that we can take $\sqrt{n} \geqslant 4$, or $n \geqslant 16$.

Hence, 16 is the smallest number of bulbs to be purchased so that the succession of light bulbs produces light for at least 47 months with probability at least 97.5% □

Note: The Central Limit Theorem is not needed in the solution of Problem 1 because the random variables, X_k's, are normally distributed and independent. Hence, their sum is also normally distributed. Thus, no approximation is needed. Furthermore, in view of the small number n that we get in the answer, the Central Limit Theorem is not even applicable to the situation in Problem 1.

2. Four hundred fair coins are tossed simultaneously. Use the Central Limit Theorem to estimate the probability that exactly 200 of the coins come up heads. Explain your reasoning.

   Solution: Let $X_1, X_2, \ldots, X_n$, where $n = 400$, denote n independent Bernoulli(1/2) trials corresponding to the outcomes of the toss of each coin ($X_k = 1$) being head, and ($X_k = 0$) being tail. Then

$$Y_n = \sum_{k=1}^{n} X_k$$

   counts the number of heads in n tosses. We would like to estimate

$$\Pr(Y_n = 200) = \Pr(199.5 < Y_n \leqslant 200.5),$$

   where we have used the continuity correction.

   Using the Central Limit Theorem, we can get the approximation

$$\Pr(Y_n = 200) \approx \Pr\left(\frac{199.5 - 200}{\sqrt{400}(1/2)} < Z \leqslant \frac{200.5 - 200}{\sqrt{400}(1/2)}\right),$$

   where $Z \sim \text{Normal}(0, 1)$. We then have that

$$\begin{aligned} \Pr(Y_n = 200) &\approx \Pr(-0.05 < Z \leqslant 0.05) \\ &\approx F_Z(0.05) - F_Z(-0.05) \\ &\approx 2F_Z(0.05) - 1; \end{aligned}$$

   so that, using a table of standard normal probabilities,

$$\Pr(Y_n = 200) \approx 2(0.5199) - 1,$$

   or

$$\Pr(Y_n = 200) \approx 0.0398.$$

   Hence, the probability that exactly 200 of the 400 coins come up heads is about 3.98%. □

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with mean λ, and set $Y_n = \sum_{i=1}^{n} X_i$, for $n = 1, 2, 3, \ldots$

   We showed in class that

$$2\sqrt{n}\left(\sqrt{\frac{Y_n}{n}} - \sqrt{\lambda}\right) \xrightarrow{D} Z \sim \text{Normal}(0, 1) \text{ as } n \to \infty. \tag{5}$$

(a) Use the fact in (5) to obtain a formula for computing an approximate 95% confidence interval for the mean λ in terms of the sample mean $\overline{X}_n = \dfrac{Y_n}{n}$.

Solution: We find a $z^* > 0$ so that, in view of (5),

$$\Pr\left(2\sqrt{n}\left|\sqrt{\lambda} - \sqrt{\overline{X}_n}\right| \leqslant z^*\right) \approx \Pr(-z^* < Z \leqslant z^*) \geqslant 0.95, \tag{6}$$

where $Z \sim \text{Normal}(0, 1)$.

Thus, we first solve for z^* in the inequality

$$F_{_Z}(z^*) - F_{_Z}(-z^*) \geqslant 0.95,$$

or

$$2F_{_Z}(z^*) - 1 \geqslant 0.95,$$

or

$$F_{_Z}(z^*) \geqslant 0.975.$$

Thus, we can take $z^* = 1.96$, which we can round up to 2. Hence, in view of (6)), an approximate 95% confidence interval for the mean λ in terms of the sample mean $\overline{X}_n$ can be computed from

$$\left|\sqrt{\lambda} - \sqrt{\overline{X}_n}\right| \leqslant \frac{1}{\sqrt{n}},$$

or

$$\sqrt{\overline{X}_n} - \frac{1}{\sqrt{n}} \leqslant \sqrt{\lambda} \leqslant \sqrt{\overline{X}_n} + \frac{1}{\sqrt{n}};$$

so that

$$\left(\sqrt{\overline{X}_n} - \frac{1}{\sqrt{n}}\right)^2 \leqslant \lambda \leqslant \left(\sqrt{\overline{X}_n} + \frac{1}{\sqrt{n}}\right)^2. \tag{7}$$

□

(b) Give an approximate 95% confidence interval for λ for a sample size of 36 and a sample mean of 25.

Solution: Use (7) with $\overline{X}_n = 25$ and $n = 36$ to get

$$\left(5 - \frac{1}{6}\right)^2 \leqslant \lambda \leqslant \left(5 + \frac{1}{6}\right)^2,$$

or

$$23.36 \leqslant \lambda \leqslant 26.69.$$

□