Solutions to Assignment #13

1. Let $W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}(2,2) \;\middle|\; d = a \text{ and } c = -b \right\}$. Prove that W is a subspace of $\mathbb{M}(2,2)$.

   Proof: First, observe that the 2×2 zero matrix $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is in W; hence, W is not empty.

   Next, let $A \in W$, then $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$; so that
$$tA = \begin{pmatrix} ta & tb \\ -tb & ta \end{pmatrix},$$
   which is also in W. Therefore, W is closed under scalar multiplication.

   To see that W is closed under matrix addition, let A_1 and A_2 be two matrices in W. Then,
$$A_1 = \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix},$$
   for scalars a_1, a_2, b_1, b_2. Then,
$$A_1 + A_2 = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ -(b_1 + b_2) & a_1 + a_2 \end{pmatrix},$$
   which is in W.

   We have seen therefore that W is non–empty, and closed under matrix addition and scalar multiplication. Hence, W is a subspace of $\mathbb{M}(2,2)$. □

2. Let W be as in Problem 1. Find a basis for W and compute $\dim(W)$.

   ***Solution**:* Given any $A \in W$, write
$$\begin{aligned} A &= \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \\ &= \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \\ &= a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \end{aligned}$$

which shows that A is in the spam of the set

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

Since the matrices are not multiples of each other, it follows that $\mathcal{B}$ is a basis for W. Hence, $\dim(W) = 2$. $\square$

3. Let $W = \{A \in \mathbb{M}(2,2) \mid A^T = A\}$; that is, W is the set of all 2×2 symmetric matrices. Prove that W is a subspace of $\mathbb{M}(2,2)$. Find a basis for W and compute its dimension.

***Solution**:* A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in W iff $c = b$. Hence,

$$\begin{aligned} A &= \begin{pmatrix} a & b \\ b & d \end{pmatrix} \\ &= \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \\ &= a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

which shows that W is the span of the set

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\};$$

hence, W is a subspace of $\mathbb{M}(2,2)$.

Next, we see that $\mathcal{B}$ is linearly independent. To see why this is so, suppose that c_1, c_2 and c_3 solve the matrix equation

$$c_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{1}$$

or

$$\begin{pmatrix} c_1 & c_2 \\ c_2 & c_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

from which we get that

$$c_1 = c_2 = c_3 = 0.$$

Thus, the matrix equation in (1) has only the trivial solution. Consequently, $\mathcal{B}$ is linearly independent.

We therefore conclude that $\mathcal{B}$ is a basis for W and so $\dim(W) = 3$.
$\square$

4. Determine whether or not the set

$$\left\{ \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -3 \\ 6 & -3 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} \right\} \tag{2}$$

forms a basis for $\mathbb{M}(2,2)$.

Solution: Denote the set in (2) by $\mathcal{S}$ and consider the matrix equation

$$c_1 \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + c_2 \begin{pmatrix} -2 & 0 \\ 3 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 & -3 \\ 6 & -3 \end{pmatrix} + c_4 \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3}$$

This leads to the system of equations

$$\left\{ \begin{array}{rcl} c_1 - 2c_2 + c_3 + c_4 & = & 0 \\ -c_1 - 3c_3 + c_4 & = & 0 \\ c_1 + 3c_2 + 6c_3 - 4c_4 & = & 0 \\ -c_1 - 3c_3 + c_4 & = & 0. \end{array} \right. \tag{4}$$

The augmented matrix of this system is:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \qquad \left(\begin{array}{rrrr|r} 1 & -2 & 1 & 1 & 0 \\ -1 & 0 & -3 & 1 & 0 \\ 1 & 3 & 6 & -4 & 0 \\ -1 & 0 & -3 & 1 & 0 \end{array} \right).$$

We can reduce this matrix to

$$\left(\begin{array}{rrrr|r} 1 & 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

Thus, system (4) is equivalent to the system

$$\left\{ \begin{array}{rcl} c_1 + 3c_3 - c_4 & = & 0 \\ c_2 + c_3 - c_4 & = & 0, \end{array} \right.$$

which has more unknowns than equations. Consequently, it has infinitely many solutions. Therefore, the matrix equation (3) has non-trivial solutions and therefore the set $\mathcal{S}$ is linearity dependent and so it cannot be a basis for $\mathbb{M}(2,2)$. □

5. Let $W = \{A \in \mathbb{M}(n,n) \mid A \text{ is a diagonal matrix}\}$; that is,

$$A = [a_{ij}] \in W \text{ iff } a_{ij} = 0 \text{ for all } \ i \neq j.$$

Prove that W is a subspace of $\mathbb{M}(n,n)$ and compute $\dim(W)$.

***Solution**:* If $A \in W$, then

$$\begin{aligned}
A &= \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \\
&= \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \\
&\qquad + \cdots + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \\
&= a_{11} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} + a_{22} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \\
&\qquad + \cdots + a_{nn} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.
\end{aligned}$$

Thus, A is in the span of the n matrices

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Labeling these matrices $A_1, A_2, \ldots, A_n$, respectively, we see that

$$W = \text{span}\{A_1, A_2, \ldots, A_n\}.$$

This shows that W is a subspace of $\mathbb{M}(n, n)$.

It is not hard to see that the set $\mathcal{B} = \{A_1, A_2, \ldots, A_n\}$ is also linearly independent. Thus, $\mathcal{B}$ is a basis for W, and therefore $\dim(W) = n$.

□