Solutions to Assignment #14

1. Let $\mathbb{C}(2,2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}(2,2) \;\middle|\; d = a \text{ and } c = -b \right\}$. It was shown in Problem 1 in Assignment #13 that $\mathbb{C}(2,2)$ is a subspace of $\mathbb{M}(2,2)$.

   (a) Prove that $\mathbb{C}(2,2) = \text{span}\{I, J\}$, where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

   Solution: Given any $A \in \mathbb{C}(2,2)$, write

$$\begin{aligned} A &= \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \\ &= \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \\ &= a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (-b)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= aI + (-b)J, \end{aligned}$$

   which shows that $A \in \text{span}\{I, J\}$. Thus,

$$\mathbb{C}(2,2) \subseteq \text{span}\{I, J\}. \tag{1}$$

   Next, since I and J are in in $\mathbb{C}(2,2)$, and $\mathbb{C}(2,2)$ is a subspace of $\mathbb{M}(2,2,)$, it follows that

$$\text{span}\{I, J\} \subseteq \mathbb{C}(2,2). \tag{2}$$

   Combining (1) and (2) yields $\mathbb{C}(2,2) = \text{span}\{I, J\}$. □

   (b) Observe that $J^2 = JJ = -I$ and compute J^n, where $n = 1, 2, 3, \ldots$

   Solution: Since matrix multiplication is associative, we can compute

$$\begin{aligned} J^3 &= (J^2)J = (-I)J = -J, \\ J^4 &= (J^3)J = (-J)(J) = -J^2 = -(-I) = I, \\ J^5 &= (J^4)J = (I)(J) = J, \end{aligned}$$

and so on. We therefore get the following pattern

$$J^n = \begin{cases} I & \text{if } n = 4k, \\ J & \text{if } n = 4k+1, \\ -I & \text{if } n = 4k+2, \\ -J & \text{if } n = 4k+3, \end{cases}$$

for $k = 1, 2, 3, \ldots$ □

2. Let $\mathbb{C}(2,2)$ be as in Problem 1.

   (a) Prove that if Z_1 and Z_2 are two matrices in $\mathbb{C}(2,2)$, then $Z_1Z_2 \in \mathbb{C}(2,2)$; that is, $\mathbb{C}(2,2)$ is closed under matrix multiplication.

   Solution: Let $Z_1 = a_1I + b_1J$ and $Z_2 = a_2I + b_2J$; then, applying the distributive and associative properties of matrix algebra,

$$\begin{aligned} Z_1Z_2 &= (a_1I + b_1J)(a_2I + b_2J) \\ &= a_1a_2I^2 + a_1b_2IJ + b_1a_2JI + b_1b_2JJ \\ &= a_1a_2I + a_1b_2J + b_1a_2J + b_1b_2J^2 \\ &= a_1a_2I + (a_1b_2 + b_1a_2)J + b_1b_2(-I) \\ &= (a_1a_2 - b_1b_2)I + (a_1b_2 + b_1a_2)J, \end{aligned}$$

   which shows that $Z_1Z_2 \in \text{span}\{I, J\}$ and therefore $Z_1Z_2 \in \mathbb{C}(2,2)$. □

   (b) Let Z_1 and Z_2 be two matrices in $\mathbb{C}(2,2)$. Prove that $Z_1Z_2 = Z_2Z_1$; that is, matrix multiplication in $\mathbb{C}(2,2)$ is commutative.

   Solution: Let Z_1 and Z_2 be as in the solution to part (a) above; then, by the calculation done in that solution

$$\begin{aligned} Z_1Z_2 &= (a_1a_2 - b_1b_2)I + (a_1b_2 + b_1a_2)J \\ &= (a_2a_1 - b_2b_1)I + (b_2a_1 + a_2b_1)J \\ &= (a_2a_1 - b_2b_1)I + (a_2b_1 + b_2a_1)J \\ &= Z_2Z_1. \end{aligned}$$

□

(c) Give the coordinates of Z_1, Z_2 and Z_1Z_2 relative to the basis $\mathcal{B} = \{I, J\}$ of $\mathbb{C}(2,2)$.

Solution: Let Z_1 and Z_2 be as in the solution to part (a) above. Then,

$$[Z_1]_{\mathcal{B}} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \ [Z_2]_{\mathcal{B}} = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \quad \text{and} \quad [Z_1Z_2]_{\mathcal{B}} = \begin{pmatrix} a_1a_2 - b_1b_2 \\ a_1b_2 + b_1a_2 \end{pmatrix}.$$

□

3. Let $\mathbb{C}(2,2)$ be as in Problem 1.

(a) Let $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where $a^2 + b^2 \neq 0$. Prove that there exists a matrix Z in $\mathbb{C}(2,2)$ such that

$$AZ = I.$$

Suggestion: Write $Z = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$, where x and y denote real numbers, compute AZ and find x and y so that $AZ = I$. Consider separately the cases $a \neq 0$ and $a = 0$. Observe that, since $a^2 + b^2 \neq 0$, if $a = 0$, then $b \neq 0$.

Solution: Assume that $a^2 + b^2 \neq 0$ and look for $Z = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$, where x and y are unknown, such that $AZ = I$, where I is the 2×2 identity matrix; that is,

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

or

$$\begin{pmatrix} ax - by & -(bx + ay) \\ bx + ay & ax - by \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This leads to a system of the two linear equations

$$\left\{ \begin{array}{rcl} ax - by & = & 1 \\ bx + ay & = & 0. \end{array} \right. \tag{3}$$

We can solve the system in (3) by performing elementary row operations on the augmented matrix

$$\begin{pmatrix} a & -b & | & 1 \\ b & a & | & 0 \end{pmatrix}. \tag{4}$$

We first reduce the matrix in (4) for the case in which $a \neq 0$. We obtain that

$$\begin{pmatrix} 1 & 0 & | & a/(a^2+b^2) \\ 0 & 1 & | & -b/(a^2+b^2) \end{pmatrix}, \tag{5}$$

where we have used the assumption that $a^2 + b^2 \neq 0$. From (5) we get that the system in (3) has the unique solution

$$x = \frac{a}{a^2+b^2} \quad \text{and} \quad y = -\frac{b}{a^2+b^2}.$$

It then follows that

$$Z = \begin{pmatrix} \dfrac{a}{a^2+b^2} & \dfrac{b}{a^2+b^2} \\ \\ -\dfrac{b}{a^2+b^2} & \dfrac{a}{a^2+b^2} \end{pmatrix},$$

or

$$Z = \frac{1}{a^2+b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}. \tag{6}$$

Next, consider the case $a = 0$. Since $a^2 + b^2 \neq 0$, it follows that $b \neq 0$. In this case, the augmented matrix in (4) becomes

$$\begin{pmatrix} 0 & -b & | & 1 \\ b & 0 & | & 0 \end{pmatrix},$$

which can be reduced to

$$\begin{pmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & -1/b \end{pmatrix},$$

Thus,

$$x = 0 \quad \text{and} \quad y = -\frac{1}{b}.$$

Contently, if $a = 0$ and $b \neq 0$, then

$$Z = \begin{pmatrix} 0 & 1/b \\ -1/b & 0 \end{pmatrix},$$

or

$$Z = \frac{1}{b} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Observe that this is the same matrix obtained from (6) by setting $a = 0$. Thus, in all cases, Z is given by (6). $\square$

(b) Put $\mathcal{B} = \{I, J\}$ and find the coordinates of A and Z relative to $\mathcal{B}$.

Solution: If $A = aI + bJ$ then

$$[A]_{\mathcal{B}} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

If $a^2 + b^2 \neq 0$, then $Z = xI + yJ$ such that $AZ = I$ is given by

$$[Z]_{\mathcal{B}} = \frac{1}{a^2+b^2}\begin{pmatrix} a \\ -b \end{pmatrix}.$$

□

4. Consider the system of linear equations

$$\begin{cases} 2x_1 - x_2 - 3x_3 &= 4 \\ x_1 + \ x_2 + x_3 &= -2 \\ x_1 + 2x_2 + 3x_3 &= 5. \end{cases} \tag{7}$$

(a) Find a 3×3 matrix A and 3×1 matrices x and b (that is, x and y are vectors in $\mathbb{R}^3$) so that the system in (7) can be expressed as the matrix equation

$$Ax = b.$$

Answer:

$$A = \begin{pmatrix} 2 & -1 & -3 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}.$$

□

(b) Let C denote the matrix $\begin{pmatrix} 1 & -3 & 2 \\ -2 & 9 & -5 \\ 1 & -5 & 3 \end{pmatrix}$, and compute the products CA, AC and Cb.

Answer:

$$AC = CA = I,$$

where I denotes the 3×3 identity matrix, and

$$Cb = \begin{pmatrix} 20 \\ -51 \\ 29 \end{pmatrix}.$$

□

(c) Prove that $x = Cb$ is the unique solution to the system in (7).

Solution: Using the fact that $AC = I$ and the associativity of matrix multiplication, we see that

$$A(Cb) = (AC)b = Ib = b,$$

so that $x = Cb$ is a solution to the equation $Ax = b$.
To see that $Ax = b$ has a unique solution, assume that there are two solutions, x and y, so that

$$Ax = Ay.$$

Subtracting Ay on both sides and using the distributive property we get that

$$A(x - y) = \mathbf{0}.$$

Multiplying by C on both sides we get that

$$C(A(x - y)) = C\mathbf{0},$$

or

$$(CA)(x - y) = \mathbf{0},$$

or

$$I(x - y) = \mathbf{0},$$

or

$$x - y = \mathbf{0},$$

from which we get that $x = y$. □

5. Find matrices A and B in $\mathbb{M}(2, 2)$ that have no entries equal to 0, but such that

$$AB = O,$$

where O denotes the 2×2 zero matrix.

Explain why, in this case, it is impossible to find 2×2 matrix C such that $CA = I$, where I denotes the 2×2 identity matrix.

Solution: Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and $B = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$. Then,

$$AB = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

If there was a 2×2 matrix C such that $CA = I$, then, multiplying on both sides of

$$AB = O$$

by C we get that

$$C(AB) = CO,$$

or

$$(CA)B = O,$$

or

$$IB = O,$$

which implies that $B = O$; but B is not the zero matrix. Consequently, there is no 2×2 matrix C such that $CA = I$. $\square$