Assignment #15

Due on Monday, November 3, 2014

Read Section 3.2, on *Matrix Algebra,* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Do the following problems

1. Let A be an $m \times n$ matrix, and $\{e_1, e_2, \ldots, e_n\}$ denote the standard basis in $\mathbb{R}^n$.
   (a) Prove that Ae_j is the j^{th} column of the matrix A.
   (b) Use your result from part (a) to prove that $AI = A$, where I denotes the $n \times n$ identity matrix.

2. Recall that the null space of a matrix $A \in \mathbb{M}(m, n)$, denoted by N_A, is the space of solutions to the equation $Ax = \mathbf{0}$; that is, $N_A = \{v \in \mathbb{R}^n \mid Av = \mathbf{0}\}$. Prove that $v \in N_A$ if and only if v is orthogonal to the rows of A.

3. Recall that the transpose of an $m \times n$ matrix, $A = [a_{ij}]$, is the $n \times m$ matrix A^T given by $A^T = [a_{ji}]$, for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$.
   Let $A \in \mathbb{M}(m, n)$ and $B \in \mathbb{M}(n, k)$. Prove that $(AB)^T = B^T A^T$.

4. Consider any diagonal matrix $A = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \in \mathbb{M}(3, 3)$.
   Prove that there exist constants c_o , c_1 , c_2 and c_3 such that
   $$c_o I + c_1 A + c_2 A^2 + c_2 A^3 = O,$$
   where I is the identity matrix in $\mathbb{M}(3, 3)$ and O denotes the 3×3 zero–matrix. In other words, there exists a polynomial, $p(x) = c_o + c_1 x + c_2 x^2 + c_3 x^3$, of degree 3, such that $p(A) = O$.

5. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & 3 \\ 4 & 1 & 2 \end{pmatrix}$.
   (a) Compute A^2 and A^3.
   (b) Verify that $A^3 - A^2 - 11A - 25I = O$, where I is the identity matrix in $\mathbb{M}(3, 3)$ and O denotes the 3×3 zero–matrix.
   (c) Use the result of part (b) above to find a matrix $B \in \mathbb{M}(3, 3)$ such that $AB = I$.