Solutions to Assignment #16

1. Define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ as follows: For each $v \in \mathbb{R}^2$, $T(v)$ is the reflection of the point determined by the coordinates of v, relative to the standard basis in $\mathbb{R}^2$, on the line $y = x$ in $\mathbb{R}^2$. That is, $T(v)$ determines a point along a line through the point determined by v which is perpendicular to the line $y = x$, and the distance from v to the line $y = x$ is the same as the distance from $T(v)$ to the line $y = x$ (see Figure 1).

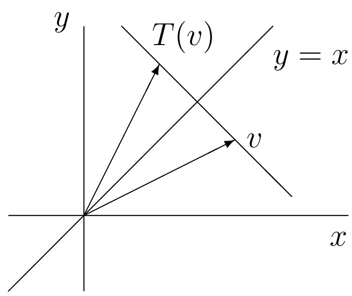

Figure 1: Reflection on the line $y = x$

Prove that T is a linear function.

Solution: If $\begin{pmatrix} x \\ y \end{pmatrix}$ denote the coordinates of v relative to the standard basis in $\mathbb{R}^2$, then
$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix},$$
which can be written as
$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$
and this shows that T is linear since T is multiplication by the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. □

2. Prove that if $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear, then $T(\mathbf{0}) = \mathbf{0}$, where the first $\mathbf{0}$ denotes the zero–vector in $\mathbb{R}^n$ and the second $\mathbf{0}$ denotes the zero–vector in $\mathbb{R}^m$.

Proof: Start with $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and apply the transformation T on both sides to get

$$T(\mathbf{0} + \mathbf{0}) = T(\mathbf{0}),$$

or

$$T(\mathbf{0}) + T(\mathbf{0}) = T(\mathbf{0}),$$

where we have used the linearity of T on the left–hand side of the equation. Thus, adding $-T(\mathbf{0})$ to bout sides of the equation yields

$$T(\mathbf{0}) = \mathbf{0}.$$

□

3. Suppose that $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear and define

$$\mathcal{N}_T = \{v \in \mathbb{R}^n \mid T(v) = \mathbf{0}\},$$

where $\mathbf{0}$ denotes the zero–vector in $\mathbb{R}^m$.

Prove that $\mathcal{N}_T$ is a subspace of $\mathbb{R}^n$.

Note: $\mathcal{N}_T$ is called the **null space** of the linear function T.

Solution: By the result of Problem 2, $T(\mathbf{0}) = \mathbf{0}$, which shows that $\mathbf{0} \in \mathcal{N}_T$ and therefore $\mathcal{N}_T$ is not empty.

Next, we show that $\mathcal{N}_T$ is closed under the vector space operations in $\mathbb{R}^n$.

Let $v \in \mathcal{N}_T$; then, $T(v) = \mathbf{0}$ and therefore

$$T(cv) = cT(v) = c\mathbf{0} = \mathbf{0}.$$

Thus, $cv \in \mathcal{N}_T$. This shows closure under scalar multiplication for $\mathcal{N}_T$.

Next, let $v, w \in \mathcal{N}_T$; then $T(v) = \mathbf{0}$ and $T(W) = \mathbf{0}$, so that

$$T(v + w) = T(v) + T(w) = \mathbf{0} + \mathbf{0} = \mathbf{0},$$

which shows that $v + w \in \mathcal{N}_T$ and therefore $\mathcal{N}_T$ is closed under vector addition. □

4. Suppose that $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear and define

$$\mathcal{I}_T = \{w \in \mathbb{R}^m \mid w = T(v) \text{ for some } v \in \mathbb{R}^n\}.$$

Prove that $\mathcal{I}_T$ is a subspace of $\mathbb{R}^m$.

Note: The set $\mathcal{I}_T$ is called the **image** of the function T. It is also denoted by $T(\mathbb{R}^n)$; thus,

$$T(\mathbb{R}^n) = \{w \in \mathbb{R}^m \mid w = T(v) \text{ for some } v \in \mathbb{R}^n\}.$$

***Solution**:* Using again the result of Problem 2 we see that $T(\mathbf{0}) = \mathbf{0}$ which shows that $\mathbf{0} \in \mathcal{I}_T$. Thus, $\mathcal{I}_T$ is nonempty.
We next verify the closure properties.
Suppose that $w \in \mathcal{I}_T$; then, there exists $v \in \mathbb{R}$ such that

$$w = T(v).$$

Multiplying by $c \in \mathbb{R}$ and using the linearity of T we obtain that

$$cw = cT(v) = T(cv),$$

which shows that $cw \in \mathcal{I}_T$ and therefore $\mathcal{I}_T$ is closed under scalar multiplication.
Next, let $w_1, w_2 \in \mathcal{I}_T$; then, there exist $v_1, v_2 \in \mathbb{R}$ such that

$$w_1 = T(v_1) \quad \text{and} \quad w_2 = T(v_2).$$

We then get that

$$w_1 + w_2 = T(v_1) + T(v_2) = T(v_1 + v_2),$$

where we have used the linearity of T. This shows that $w_1 + w_2 \in \mathcal{I}_T$ and therefore $\mathcal{I}_T$ is closed under vector addition. □

5. Fix $u \in \mathbb{R}^n$ and define $f\colon \mathbb{R}^n \to \mathbb{R}$ by

$$f(v) = \langle u, v\rangle \quad \text{for all} \quad v \in \mathbb{R}^n.$$

   (a) Prove that f is a linear function.

Solution: The linearity of f follows by the bi–linearity of the Euclidean inner–product:

$$\langle u, cv + dw\rangle = c\langle u, v\rangle + d\langle u, w\rangle. \tag{1}$$

Taking $d = 0$ in (1) yields

$$f(cv) = cf(v),$$

for all scalars c and vectors $v \in \mathbb{R}^n$, and taking $c = d = 1$ in (1) yields

$$f(v + w) = f(v) + f(w)$$

for all $v, w \in \mathbb{R}^n$. □

(b) Let $\mathcal{N}_f$ denote the null space of f; that is,

$$\mathcal{N}_f = \{v \in \mathbb{R}^n \mid \langle u, v\rangle = 0\}.$$

Find the dimension of $\mathcal{N}_f$ for each of the cases: $u = \mathbf{0}$ and $u \neq \mathbf{0}$.

Solution: Suppose first that $u = \mathbf{0}$. Then, $\mathcal{N}_f = \mathbb{R}^n$ since $\langle \mathbf{0}, v\rangle = 0$ for all $v \in \mathbb{R}^n$. Thus, in this case,

$$\dim(\mathcal{N}_f) = n.$$

Next, suppose that $u \neq \mathbf{0}$. Let $\{w_1, w_2, \ldots, w_k\}$ denote a basis for $\mathcal{N}_f$ so that $k = \dim(\mathcal{N}_f)$. Observe that, for any $v \in \mathbb{R}^n$, the vector

$$v - \frac{\langle u, v\rangle}{\|u\|^2}\, u,$$

is in $\mathcal{N}_f$. To see why this is the case, compute

$$\begin{aligned}
\left\langle u, v - \frac{\langle u, v\rangle}{\|u\|^2}\, u \right\rangle &= \langle u, v\rangle - \frac{\langle u, v\rangle}{\|u\|^2}\langle u, u\rangle \\
&= \langle u, v\rangle - \frac{\langle u, v\rangle}{\|u\|^2}\|u\|^2 \\
&= \langle u, v\rangle - \langle u, v\rangle \\
&= 0.
\end{aligned}$$

It then follows that the vector $v - \dfrac{\langle u, v\rangle}{\|u\|^2}\, u$ is a linear combination of the vectors in $\{w_1, w_2, \ldots, w_k\}$; that is,

$$v - \frac{\langle u, v\rangle}{\|u\|^2}\, u = c_1 w_1 + c_2 w_2 + \cdots + c_k w_k,$$

so that

$$v = \frac{\langle u, v\rangle}{\|u\|^2}\, u + c_1 w_1 + c_2 w_2 + \cdots + c_k w_k.$$

Hence,

$$\mathbb{R}^n = \text{span}\{u, w_1, w_2, \ldots, w_k\}$$

Next, we see that the set $\{u, w_1, w_2, \ldots, w_k\}$ is linearly independent. To show this, let $d, c_1, c_2, \ldots, c_k$ be any solution of the vector equation

$$du + c_1 w_1 + c_2 w_2 + \cdots + c_k w_k = \mathbf{0}. \tag{2}$$

Apply f to both sides of (2) and use the linearity of f and the result of Problem (2) to get that

$$df(u) + c_1 f(w_1) + c_2 f(w_2) + \cdots + c_k f(w_k) = 0,$$

or

$$d\|u\|^2 = 0,$$

since $w_1, w_2, \ldots, w_k \in \mathcal{N}_f$. We therefore conclude that $d = 0$ because $u \neq \mathbf{0}$. Hence, (2) becomes

$$c_1 w_1 + c_2 w_2 + \cdots + c_k w_k = \mathbf{0},$$

which implies that $c_1 = c_2 = \cdots = c_k = 0$, since the set

$$\{w_1, w_2, \ldots, w_k\}$$

is linearly independent. We have therefore shown that the vector equation in (2) has only the trivial solution and therefore the set $\{u, w_1, w_2, \ldots, w_k\}$ is linearly independent. Since this set also spans $\mathbb{R}^n$, $\{u, w_1, w_2, \ldots, w_k\}$ is a basis for $\mathbb{R}^n$ and therefore

$$1 + k = n,$$

from which we get that $k = n - 1$, or

$$\dim(\mathcal{N}_f) = n - 1$$

in the case $u \neq \mathbf{0}$. □