## Assignment #17

**Due on Wednesday, November 12, 2014**

**Read** Section 4.3 on *Matrix Representation of Linear Functions* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

**Read** Section 2.2 on *Linear Transformations between Finite Dimensional Vector Spaces* in Damiano and Little (pp. 73–81)

**Read** Section 2.3 on *Kernel and Image* in Damiano and Little (pp. 84–92)

**Do** the following problems

1. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a function satisfying

$$f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \; f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \quad \text{and} \quad f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

   (a) Show that $f$ cannot be linear.

   (b) What would $f\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ be if $f$ was a linear function?

2. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear function satisfying

$$T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \quad \text{and} \quad T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \\ 1 \end{pmatrix}.$$

   (a) Find the matrix representation for $T$ relative to the standard bases in $\mathbb{R}^2$ and $\mathbb{R}^3$.

   (b) Give formula for computing $T\begin{pmatrix} x \\ y \end{pmatrix}$ for any $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$.

   (c) Compute $T\begin{pmatrix} 4 \\ 7 \end{pmatrix}$.

3. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ denote the linear transformation defined in Problem **??**.

   (a) Determine the image, $\mathcal{I}_T = \{w \in \mathbb{R}^3 \mid w = T(v) \text{ for some } v \in \mathbb{R}^2\}$, of $T$.

   (b) Find a basis for $\mathcal{I}_T$ and compute $\dim(\mathcal{I}_T)$.

4. The projection $P_u\colon \mathbb{R}^3 \to \mathbb{R}^3$ onto the direction of the unit vector $u$ in $\mathbb{R}^3$ is given by
$$P_u(v) = \langle v, u\rangle\, u \quad \text{for all } v \in \mathbb{R}^3,$$
where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product in $\mathbb{R}^3$. We proved in class that $P_u$ is a linear function.

   (a) For $u = \dfrac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$, give the matrix representation for $P_u$ relative to the standard basis in $\mathbb{R}^3$.

   (b) For $u$ as defined in the previous part, determine the null space,
$$\mathcal{N}_{P_u} = \{v \in \mathbb{R}^3 \mid P_u(v) = \mathbf{0}\},$$
of $P_u$.

   (c) Find a basis for $\mathcal{N}_{P_u}$ and compute $\dim(\mathcal{N}_{P_u})$.

5. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ and $R\colon \mathbb{R}^m \to \mathbb{R}^k$ denote two linear functions. The composition of $R$ and $T$, denoted by $R \circ T$, is the function $R \circ T\colon \mathbb{R}^n \to \mathbb{R}^k$ defined by
$$R \circ T(v) = R(T(v)) \quad \text{for all } \; v \in \mathbb{R}^n.$$

   (a) Prove that the composition $R \circ T$ is a linear function from $\mathbb{R}^n$ to $\mathbb{R}^k$.

   (b) Show that $\mathcal{N}_T \subseteq \mathcal{N}_{R\circ T}$.

   (c) Show that $\mathcal{I}_{R\circ T} \subseteq \mathcal{I}_R$.