Assignment #19

Due on Monday, November 17, 2014

Read Section 4.3 on *Matrix Representation of Linear Functions* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 4.4, on *Compositions,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 2.6 on *The Inverse of a Linear Transformation* in Damiano and Little (pp. 114–120)

Background and Definitions

- **One–to–One Functions**. A function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is said to be injective, or one–to–one, if and only if $f(v) = f(u)$ implies that $v = u$.
- **Onto Functions**. A function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is said to be surjective, or onto, if and only if for every $w \in \mathbb{R}^m$, there exists a vector $v \in \mathbb{R}^n$ such that $f(v) = w$.
- **Invertible Functions**. A function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is said to be bijective, or invertible, if and only if f is one–to–one and onto.

  If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is invertible, we can define the inverse function $f^{-1}\colon \mathbb{R}^m \to \mathbb{R}^n$ by means of $f^{-1}(w) = v$ if and only if $w = f(v)$.

Do the following problems

1. Assume that $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear. Prove that T is one–to–one if and only if $\mathcal{N}_T = \{\mathbf{0}\}$, where $\mathcal{N}_T$ denotes the null space. or kernel, of T

2. Assume that $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear, and let M_T denote the matrix representation of T relative to the standard bases $\mathcal{E}_n$ and $\mathcal{E}_m$ of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

   Prove that T is one–to–one if and only if the columns of M_T are linearly independent in $\mathbb{R}^m$.

3. Assume that $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear, and let M_T denote the matrix representation of T relative to the standard bases $\mathcal{E}_n$ and $\mathcal{E}_m$ of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

   Prove that T is onto if and only if the columns of M_T span $\mathbb{R}^m$.

4. Assume that $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear. Prove that if T is invertible, then the inverse function $T^{-1}\colon \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation.

5. Assume that $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear. Prove that if T is invertible, then $m = n$.