Solutions to Assignment #19

1. Assume that $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear. Prove that T is one–to–one if and only if $\mathcal{N}_T = \{\mathbf{0}\}$, where $\mathcal{N}_T$ denotes the null space. or kernel, of T

   ***Solution**:* Assume that $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear and that T is one–to–one. Let $v \in \mathcal{N}_T$; then, $T(v) = \mathbf{0}$. Now, $T(\mathbf{0}) = \mathbf{0}$, since T is linear. Thus,
   $$T(v) = T(\mathbf{0}).$$
   Hence, since T is one–to–one, we obtain that $v = \mathbf{0}$. Therefore, $\mathcal{N}_T = \{\mathbf{0}\}$.

   Conversely, assume that $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear and that $\mathcal{N}_T = \{\mathbf{0}\}$. Suppose that
   $$T(v) = T(u);$$
   then, using the linearity of T,
   $$T(v) - T(u) = \mathbf{0},$$
   or
   $$T(v - u) = \mathbf{0},$$
   which shows that $v - u \in \mathcal{N}_T$. Thus, since $\mathcal{N}_T = \{\mathbf{0}\}$,
   $$v - u = \mathbf{0},$$
   from which we get that
   $$v = u.$$
   We have therefore shown that
   $$T(v) = T(u) \text{ implies that } v = u;$$
   that is, T is one–to–one. □

2. Assume that $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear, and let M_T denote the matrix representation of T relative to the standard bases $\mathcal{E}_n$ and $\mathcal{E}_m$ of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

   Prove that T is one–to–one if and only if the columns of M_T are linearly independent in $\mathbb{R}^m$.

Solution: By the result in Problem 1, T is one–to–one if and only if $\mathcal{N}_T = \{\mathbf{0}\}$.

Write

$$M_T = [v_1 \quad v_2 \quad \cdots \quad v_n],$$

where $v_1, v_2, \ldots, v_n$ are the columns of M_T, and consider the vector equation

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = \mathbf{0}, \tag{1}$$

where $\mathbf{0}$ is the zero–vector in $\mathbb{R}^m$. Note that the equation in (1) can be written as

$$[v_1 \quad v_2 \quad \cdots \quad v_n] \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{0},$$

or

$$M_T \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{0},$$

or

$$T \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathbf{0}.$$

Thus, any solution of (1) must be in the null–space of T. Hence, (1) has only the trivial solution if and only if $\mathcal{N}_T = \{\mathbf{0}\}$. We have therefore shown that T is one–to–one if and only if the columns of M_T are linearly independent. □

3. Assume that $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear, and let M_T denote the matrix representation of T relative to the standard bases $\mathcal{E}_n$ and $\mathcal{E}_m$ of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

   Prove that T is onto if and only if the columns of M_T span $\mathbb{R}^m$.

   Solution: Assume that T is onto. Then, given any $w \in \mathbb{R}^m$, there exists $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$ such that

$$T\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = w,$$

or

$$M_T\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = w,$$

or

$$x_1v_1 + x_2v_2 + \cdots + x_nv_n = w,$$

which shows that $w \in \text{span}(\{v_1, v_2, \ldots, v_n\})$. Hence, the set $\{v_1, v_2, \ldots, v_n\}$ of columns of M_T spans $\mathbb{R}^m$.

Conversely, suppose that $\text{span}(\{v_1, v_2, \ldots, v_n\}) = \mathbb{R}^m$. Then, given any $w \in \mathbb{R}^m$, there exists scalars $c_1, c_2, \ldots, c_n$ such that

$$w = c_1v_1 + c_2v_2 + \cdots + c_nv_n,$$

or

$$w = M_T\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix},$$

or

$$w = T\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

Thus, setting $v = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$, we have that $w = Tv$. Hence, for every $w \in \mathbb{R}^m$ there exists $v \in \mathbb{R}^n$ such that $w = T(v)$; that is, T is onto.
□

4. Assume that $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear. Prove that if T is invertible, then the inverse function $T^{-1}\colon \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation.

   Solution: Let w_1 and w_2 be vectors in $\mathbb{R}^m$ and put $v_1 = T^{-1}(w_1)$ and $v_2 = T^{-1}(w_2)$. Then, $v_1, v_2 \in \mathbb{R}^n$ and

   $$T(v_1) = w_1 \quad \text{and} \quad T(v_2) = w_2.$$

   Then, since T is linear,

   $$T(v_1 + v_2) = T(v_1) + T(v_2),$$

   or

   $$T(v_1 + v_2) = w_1 + w_2. \tag{2}$$

   It follows from (2) and the definition of T^{-1} that

   $$T^{-1}(w_1 + w_2) = v_1 + v_2,$$

   or

   $$T^{-1}(w_1 + w_2) = T^{-1}(w_1) + T^{-1}(w_2). \tag{3}$$

   Next, let $w \in \mathbb{R}^m$ and $c \in \mathbb{R}$. Put $v = T^{-1}(w$. Then, $v \in \mathbb{R}^n$ and $T(v) = w$.
   Now, since T is linear

   $$T(cv) = cT(v),$$

   or

   $$T(cv) = cw. \tag{4}$$

   It follows from (4) and the definition of T^{-1} that

   $$T^{-1}(cw) = cv,$$

   or

   $$T^{-1}(cw) = cT^{-1}(w). \tag{5}$$

   The results in (3) and (5) establish the linearity of T^{-1}. □

5. Assume that $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear. Prove that if T is invertible, then $m = n$.

***Solution**:* Assume that $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear and invertible. Then, T is one–to–one and onto. It then follows from the results in Problem 2 and Problem 3, respectively, that the columns of

$$M_T = [T(e_1) \quad T(e_2) \quad \cdots \quad T(e_n)]$$

are linearly independent and span $\mathbb{R}^m$. Hence, the set

$$\{T(e_1), T(e_2), \ldots, T(e_n)\}$$

is a basis for $\mathbb{R}^m$. Consequently, $\dim(\mathbb{R}^m) = n$, from which we get that $m = n$. $\square$