Assignment #20

Due on Friday, November 21, 2014

Read Section 4.3 on *Matrix Representation of Linear Functions* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 4.4, on *Compositions,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 2.3 on *Kernel and Image* in Damiano and Little (pp. 84–92)

Read Section 2.4 on *Applications of the Dimension Theorem* in Damiano and Little (pp. 95–103)

Background and Definitions

- **Null Space of a Linear Transformation**. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ denote a linear transformation. The null space of T, denoted by $\mathcal{N}_T$, is the set

  $$\mathcal{N}_T = \{v \in \mathbb{R}^n \mid T(v) = \mathbf{0}\}.$$

  Note: In the text, the null space of T is called the kernel of T, and is denoted by $\text{Ker}(T)$.

- **Image**. Given a function $T\colon \mathbb{R}^n \to \mathbb{R}^m$, the image of T, denoted by $\mathcal{I}_T$, is the set

  $$\mathcal{I}_T = \{w \in \mathbb{R}^m \mid w = T(v), \text{ for some } v \in \mathbb{R}^n\}.$$

  Note: In the text, the image of T is denoted by $\text{Im}(T)$.

  If T is linear, then $\mathcal{I}_T$ is a subspace of $\mathbb{R}^m$.

- **The Dimension Theorem**. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then,

  $$\dim(\mathcal{N}_T) + \dim(\mathcal{I}_T) = n. \qquad (1)$$

Do the following problems

1. In this problem and problems (2) and (3) you will be proving the Dimension Theorem as stated in (1).

   Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Show that if $\mathcal{N}_T = \mathbb{R}^n$, then T must be the zero transformation. What is $\mathcal{I}_T$ in this case? Verify that (1) holds true in this case.

2. Suppose that $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation that is not the zero function. Put $k = \dim(\mathcal{N}_T)$.

   (a) Explain why $k < n$.

   (b) Let $\{w_1, w_2, \ldots, w_k\}$ be a basis for $\mathcal{N}_T$. Show that there exist vectors $v_1, v_2, \ldots, v_r$ in $\mathbb{R}^n$ such that $\{w_1, w_2, \ldots, w_k, v_1, v_2, \ldots, v_r\}$ is a basis for $\mathbb{R}^n$. What is r in terms of n and k?

3. Let T, $w_1, w_2, \ldots, w_k$ and $v_1, v_2, \ldots, v_r$ be as in Problem 2.

   (a) Show that the set $\{T(v_1), T(v_2), \ldots, T(v_r)\}$ is a basis for $\mathcal{I}_T$, the image of T.

   (b) Prove the Dimension Theorem.

4. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

   (a) Prove that T is one–to–one if and only if $\dim(\mathcal{I}_T) = n$.

   (b) Prove that T is onto if and only if $\dim(\mathcal{I}_T) = m$.

5. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be given by

   $$T(v) = Av, \qquad \text{for all } v \in \mathbb{R}^3,$$

   where A is the 3×3 matrix given by

   $$A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}.$$

   Determine whether or not T is

   (a) one–to–one;

   (b) onto;

   (c) invertible.