Solutions to Assignment #20

1. In this problem and problems (2) and (3) you will be proving the Dimension Theorem
$$\dim(\mathcal{N}_T) + \dim(\mathcal{I}_T) = n, \tag{1}$$
for a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$.

   Show that if $\mathcal{N}_T = \mathbb{R}^n$, then T must be the zero transformation. What is $\mathcal{I}_T$ in this case? Verify that (1) holds true in this case.

   ***Solution**:* Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ satisfying $\mathcal{N}_T = \mathbb{R}^n$. Then, $T(v) = \mathbf{0}$ for all $v \in \mathbb{R}^n$, which shows that T is the zero transformation.

   Also, since $T(v) = \mathbf{0}$ for all $v \in \mathbb{R}^n$, it follows that $\mathcal{I}_T = \{\mathbf{0}\}$.

   Hence, $\dim(\mathcal{N}_T) = n$ and $\dim(\mathcal{I}_T) = 0$. It then follows that
$$\dim(\mathcal{N}_T) + \dim(\mathcal{I}_T) = n + 0 = n,$$
and so the Dimension Theorem (1) holds true in this case. □

2. Suppose that $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation that is not the zero function. Put $k = \dim(\mathcal{N}_T)$.

   (a) Explain why $k < n$.

   ***Solution**:* If $\dim(\mathcal{N}_T) = n$, then $\mathcal{N}_T = \mathbb{R}^n$, and, therefore,
$$T(v) = \mathbf{0}, \quad \text{for all } v \in \mathbb{R}^n.$$
However, we are assuming that T is not the zero function. Hence, $\dim(\mathcal{N}_T) < n$. □

   (b) Let $\{w_1, w_2, \ldots, w_k\}$ be a basis for $\mathcal{N}_T$. Show that there exist vectors $v_1, v_2, \ldots, v_r$ in $\mathbb{R}^n$ such that $\{w_1, w_2, \ldots, w_k, v_1, v_2, \ldots, v_r\}$ is a basis for $\mathbb{R}^n$. What is r in terms of n and k?

   ***Solution**:* Let $\{w_1, w_2, \ldots, w_k\}$ be a basis for $\mathcal{N}_T$. Then $k < n$ by the result in part (a). Thus, there exists $v_1 \in \mathbb{R}^n$ such that $v_2 \notin \operatorname{span}(\{w_1, w_2, \ldots, w_k\})$. We then have that the set
$$\{w_1, w_2, \ldots, w_k, v_1\}$$
is linearly independent.

We consider two possibilities: Either (i) $\text{span}(\{w_1, w_2, \ldots, w_k, v_1\}) = \mathbb{R}^n$, or (ii) $\text{span}(\{w_1, w_2, \ldots, w_k, v_1\}) \neq \mathbb{R}^n$.
If $\text{span}(\{w_1, w_2, \ldots, w_k, v_1\}) = \mathbb{R}^n$, then $\{w_1, w_2, \ldots, w_k, v_1\}$ is a basis for $\mathbb{R}^n$ and $n = k + 1$. If not, there exists $v_2 \in \mathbb{R}^n$ such that

$$v_2 \notin \text{span}(\{w_1, w_2, \ldots, w_k, v_1\}).$$

It then follows that the set

$$\{w_1, w_2, \ldots, w_k, v_1, v_2\}$$

is linearly independent.
Again, we consider two cases: Either (i) $\text{span}(\{w_1, w_2, \ldots, w_k, v_1, v_2\}) = \mathbb{R}^n$, or (ii) $\text{span}(\{w_1, w_2, \ldots, w_k, v_1, v_2\}) \neq \mathbb{R}^n$.
If $\text{span}(\{w_1, w_2, \ldots, w_k, v_1, v_2\}) = \mathbb{R}^n$, then $\{w_1, w_2, \ldots, w_k, v_1, v_2\}$ is a basis for $\mathbb{R}^n$ and $n = k + 2$. If not, there exists $v_3 \in \mathbb{R}^n$ such that

$$v_3 \notin \text{span}(\{w_1, w_2, \ldots, w_k, v_1, v_2\}).$$

We continue in this fashion until we get vectors $v_1, v_2, \ldots, v_r$ in $\mathbb{R}^n$ such that the set

$$\{w_1, w_2, \ldots, w_k, v_1, v_2, \ldots, v_r\} \text{ is linearly independent} \tag{2}$$

and

$$\text{span}(\{w_1, w_2, \ldots, w_k, v_1, v_2, \ldots, v_r\}) = \mathbb{R}^n. \tag{3}$$

It follows from (2) and (3) that $\{w_1, w_2, \ldots, w_k, v_1, v_2, \ldots, v_r\}$ is a basis for $\mathbb{R}^n$ and therefore $k + r = n$, from which we get that $r = n - k$. □

3. Let T, $w_1, w_2, \ldots, w_k$ and $v_1, v_2, \ldots, v_r$ be as in Problem 2.

   (a) Show that the set $\{T(v_1), T(v_2), \ldots, T(v_r)\}$ is a basis for $\mathcal{I}_T$, the image of T.

   ***Solution**:* Let $v \in \mathbb{R}^n$. Then, since $\{w_1, w_2, \ldots, w_k, v_1, v_2, \ldots, v_r\}$ is a basis for $\mathbb{R}^n$, by the result in Problem 2, there exist scalars $c_1, c_2, \ldots, c_k$ and $d_1, d_2, \ldots, d_r$, such that

$$v = c_1 w_1 + c_2 w_2 + \cdots c_k w_k + d_1 v_1 + d_2 v_2 + \cdots + d_r v_r. \tag{4}$$

   Next, apply T on both sides of (4) and use the linearity of T to get

$$T(v) = c_1 T(w_1) + c_2 T(w_2) + \cdots c_k T(w_k) + d_1 T(v_1) + d_2 T(v_2) + \cdots + d_r T(v_r),$$

so that

$$T(v) = d_1T(v_1) + d_2T(v_2) + \cdots + d_rT(v_r), \tag{5}$$

since $w_1, w_2, \ldots, w_k \in \mathcal{N}_T$.

Now, it follows from (5) that

$$T(v) \in \operatorname{span}(\{T(v_1), T(v_2), \ldots, T(v_r)\}), \quad \text{for all } v \in \mathbb{R}^n;$$

consequently,

$$\mathcal{I}_T \subseteq \operatorname{span}(\{T(v_1), T(v_2), \ldots, T(v_r)\}). \tag{6}$$

On the other hand, since $\mathcal{I}_T$ is a subspace of $\mathbb{R}^m$, it follows that

$$\operatorname{span}(\{T(v_1), T(v_2), \ldots, T(v_r)\}) \subseteq \mathcal{I}_T. \tag{7}$$

Combining (6) and (7) yields

$$\mathcal{I}_T = \operatorname{span}(\{T(v_1), T(v_2), \ldots, T(v_r)\}),$$

which shows that $\{T(v_1), T(v_2), \ldots, T(v_r)\}$ spans $\mathcal{I}_T$.

Next, we show that $\{T(v_1), T(v_2), \ldots, T(v_r)\}$ is linearly independent.

Consider the equation

$$c_1T(v_1) + c_2T(v_2) + \cdots + c_rT(v_r) = \mathbf{0}, \tag{8}$$

which, using the linearity of T, can be written as

$$T(c_1v_1 + c_2v_2 + \cdots + c_rv_r) = \mathbf{0} \tag{9}$$

It follows from (9) that $c_1v_1 + c_2v_2 + \cdots + c_rv_r \in \mathcal{N}_T$; so that, there exist scalars $d_1, d_2, \ldots, d_k$ such that

$$c_1v_1 + c_2v_2 + \cdots + c_rv_r = d_1w_1 + d_2w_2 + \cdots + d_kw_k,$$

which can be rewritten as

$$-d_1w_1 - d_2w_2 - \cdots - d_kw_k + c_1v_1 + c_2v_2 + \cdots + c_rv_r = \mathbf{0}. \tag{10}$$

It follows from (10) and the fact that $\{w_1, w_2, \ldots, w_k, v_1, v_2, \ldots, v_r\}$ is a basis for $\mathbb{R}^n$ that

$$c_1 = c_2 = \cdots = c_k = 0,$$

which shows that (8) has only the trivial solution. Hence, the set $\{T(v_1), T(v_2), \ldots, T(v_r)\}$ is linearly independent.

We have therefore shown that $\{T(v_1), T(v_2), \ldots, T(v_r)\}$ is a basis for $\mathcal{I}_T$.

□

(b) Prove the Dimension Theorem.

***Solution**:* It follows from the result in part (a) that $\dim(\mathcal{I}_T) = r$. Using the result in part (b) of Problem 2 that

$$r = n - k,$$

where $k = \dim(\mathcal{N}_T)$. We then have that

$$\dim(\mathcal{I}_T) = n - \dim(\mathcal{N}_T),$$

from which we get that

$$\dim(\mathcal{N}_T) + \dim(\mathcal{I}_T) = n,$$

which is the Dimension Theorem (1). □

4. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

(a) Prove that T is one–to–one if and only if $\dim(\mathcal{I}_T) = n$.

***Solution**:* It follows from the result of Problem 1 in Assignment #19 that T is one–to–one if and only if $\mathcal{N}_T = \{\mathbf{0}\}$; so that, $\dim(\mathcal{N}_T) = 0$. Consequently, it follows from the Dimension Theorem in (1) that T is one–to–one if and only if $\dim(\mathcal{I}_T) = n$. □

(b) Prove that T is onto if and only if $\dim(\mathcal{I}_T) = m$.

***Solution**:* It follows from the result of Problem 3 in Assignment #19 that T is onto if and only if $\operatorname{span}(\{T(e_1), T(e_2), \ldots, T(e_n)\}) = \mathbb{R}^m$. Thus, since

$$\operatorname{span}(\{T(e_1), T(e_2), \ldots, T(e_n)\}) = \mathcal{I}_T,$$

$\dim(\mathcal{I}_T) = m$. □

5. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T(v) = Av, \quad \text{for all } v \in \mathbb{R}^3,$$

where A is the 3×3 matrix given by

$$A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}.$$

Determine whether or not T is

(a) one–to–one;

(b) onto;

(c) invertible.

***Solution**:* First, we compute the null space of T by solving the equation

$$\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix};$$

this is equivalent to solving the homogeneous system of equations

$$\left\{ \begin{array}{rcl} x_2 + x_3 & = & 0 \\ -x_1 \quad + x_3 & = & 0 \\ -x_1 - x_2 \quad & = & 0 \end{array} \right. \tag{11}$$

The system in (11) can be solved by reducing the augmented matrix

$$\begin{pmatrix} 0 & 1 & 1 & | & 0 \\ -1 & 0 & 1 & | & 0 \\ -1 & -1 & 0 & | & 0 \end{pmatrix},$$

to

$$\begin{pmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{pmatrix},$$

which shows that the system in (11) has only the trivial solution. Hence,

$$\mathcal{N}_T = \{\mathbf{0}\},$$

and therefore

(a) T is one–to–one.

(b) Next, use the Dimension Theorem in (1) to get that $\dim(\mathcal{I}_T) = 3$, which shows that $\mathcal{I}_T = \mathbb{R}^3$, and therefore T is onto.

(c) Finally, since T is one–to–one and onto, we get that T is invertible.

□