Assignment #22

Due on Wednesday, November 26, 2014

Read Chapter 5, on *The Eigenvalue Problem,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 4.1 on *Eigenvalues and Eigenvectors* in Damiano and Little (pp. 162–173)

Read Section 4.2 on *Diagonalizability* in Damiano and Little (pp. 175–181)

Background and Definitions

Eigenvalues and Eigenvectors. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. A scalar, λ, is said to be an **eigenvalue** of T if and only if the equation

$$T(v) = \lambda v \tag{1}$$

has a nontrivial solution.

A nontrivial solution, v, of the equation $T(v) = \lambda v$ is called an **eigenvector** corresponding to the eigenvalue λ.

Observe that the equation in (1) can also be written as

$$(T - \lambda I)v = \mathbf{0},$$

where $I\colon \mathbb{R}^n \to \mathbb{R}^n$ denotes the identity transformation in $\mathbb{R}^n$. Thus, λ is an eigenvalue of T if and only if the null space of the linear transformation $T - \lambda I$ is nontrivial; that is $\mathcal{N}_{T-\lambda I} \neq \{\mathbf{0}\}$. The null space of $T - \lambda I$ is called the **eigenspace** of T corresponding to λ and is denoted by $E_T(\lambda)$.

Do the following problems

1. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote a linear transformation in $\mathbb{R}^2$. Suppose that v_1 and v_2 are two eigenvectors of T corresponding to the eigenvalues λ_1 and λ_2, respectively.

   Prove that, if $\lambda_1 \neq \lambda_2$, then the set $\{v_1, v_2\}$ is linearly independent.

   Deduce therefore that a linear transformation, T, from $\mathbb{R}^2$ to $\mathbb{R}^2$ cannot have more than two distinct eigenvalues.

2. Show that the rotation $R_\theta \colon \mathbb{R}^2 \to \mathbb{R}^2$ does not have any real eigenvalues unless $\theta = 0$ or $\theta = \pi$.

   Give the eigenvalues and corresponding eigenspaces in each case.

3. Let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{for all} \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,$$

   where A is the 2×2 matrix

$$A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}.$$

   Find all the eigenvalue of T and compute their respective eigenspaces.

4. Let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{for all} \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,$$

   where A is the 2×2 matrix

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

   where a and b are real constants.

   (a) Show that T has real eigenvalues.

   (b) Under what conditions on a and b will the eigenvalues obtained in part (a) be distinct eigenvalues?

5. Let $T \colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Prove that $\lambda = 0$ is an eigenvalue of T if and only if the matrix representation, M_T, of T is not one–to–one.