Solutions to Assignment #22

1. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote a linear transformation in $\mathbb{R}^2$. Suppose that v_1 and v_2 are two eigenvectors of T corresponding to the eigenvalues λ_1 and λ_2, respectively.

   Prove that, if $\lambda_1 \neq \lambda_2$, then the set $\{v_1, v_2\}$ is linearly independent.

   Deduce therefore that a linear transformation, T, from $\mathbb{R}^2$ to $\mathbb{R}^2$ cannot have more than two distinct eigenvalues.

   Proof: Let λ_1 and λ_2 be distinct eigenvalues of $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ with corresponding eigenvectors v_1 and v_2, respectively.

   Suppose that c_1 and c_2 solve the vector equation

   $$c_1 v_1 + c_2 v_2 = \mathbf{0}. \tag{1}$$

   Applying T on both sides of the equation in (1) and using the linearity of T, we obtain that

   $$c_1 T(v_1) + c_2 T(v_2) = \mathbf{0},$$

   or

   $$c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 = \mathbf{0}, \tag{2}$$

   since v_1 and v_2 are eigenvectors corresponding to λ_1 and λ_2, respectively.

   Since, we are assuming that λ_1 and λ_2 are distinct, one of them cannot be 0. Thus, suppose that $\lambda_2 \neq 0$ and multiply the vector equation in (1) by λ_2 to get

   $$c_1 \lambda_2 v_1 + c_2 \lambda_2 v_2 = \mathbf{0}. \tag{3}$$

   Subtracting the vector equation in (1) from the vector equation in (3) we then get that

   $$c_1(\lambda_2 - \lambda_1) v_1 = \mathbf{0},$$

   which implies that

   $$c_1 v_1 = \mathbf{0}$$

   because $\lambda_1 \neq \lambda_2$. It then follows that $c_1 = 0$ since v_1 is not the zero vector in $\mathbb{R}^2$. We then get from (1) that

   $$c_2 v_2 = \mathbf{0},$$

   which implies that $c_2 = 0$ since v_2 is not the zero vector in $\mathbb{R}^2$.

We have therefore shown that $c_1 = c_2 = 0$ is the only solution of the vector equation in (1). Consequently, $\{v_1, v_2\}$ is linearly independent.

Thus, a linear transformation, T, from $\mathbb{R}^2$ to $\mathbb{R}^2$ cannot have more than two distinct eigenvalues. For if it did, then a similar argument to the one given above would imply that there there is a set of more than two linearly independent vectors, which is impossible in $\mathbb{R}^2$ because $\dim(\mathbb{R}^2) = 2$. □

2. Show that the rotation $R_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ does not have any real eigenvalues unless $\theta = 0$ or $\theta = \pi$.

Give the eigenvalues and corresponding eigenspaces in each case.

***Solution**:* Consider the matrix for the transformation $R_\theta - \lambda I$, where I denotes the 2×2 identity matrix,

$$M_{R_\theta - \lambda I} = \begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix}.$$

The homogeneous system $(R_\theta - \lambda I)v = \mathbf{0}$ has nontrivial solutions if and only if the columns of $M_{R_\theta - \lambda I}$ are linearly dependent, and this is the case if and only if $\det(M_{R_\theta - \lambda I}) = 0$, or

$$(\cos\theta - \lambda)^2 + \sin^2\theta = 0,$$

which implies that

$$\lambda^2 - 2\cos\theta\ \lambda + 1 = 0. \tag{4}$$

The quadratic equation in (4) has real solutions if and only if

$$4\cos^2\theta - 4 \geqslant 0,$$

or

$$\cos^2\theta \geqslant 1.$$

But, $\cos^2\theta \leqslant 1$. We therefore get that

$$\cos^2\theta = 1.$$

Thus, either $\cos\theta = 1$, which yields $\theta = 0$ or 2π, or $\cos\theta = -1$, which yields $\theta = \pi$ or $-\pi$.

If $\theta = 0$, then the matrix for R_θ is

$$M_{R_0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

the identity matrix. Thus, $\lambda = 1$ is the only eigenvalue of the R_0 and every nonzero vector, v, in $\mathbb{R}^2$ is an eigenvector since $R_0 v = v$. Hence $E_{R_0}(1) = \mathbb{R}^2$.

If $\theta = \pi$, the matrix for R_θ is

$$M_{R_\pi} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

so that $\lambda = -1$ is the only eigenvalue of the R_π and every nonzero vector, v, in $\mathbb{R}^2$ is an eigenvector since $R_\pi v = -v$. Hence $E_{R_0}(-1) = \mathbb{R}^2$. $\square$

3. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{for all} \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,$$

where A is the 2×2 matrix

$$A = \begin{pmatrix} 5 & 3 \\ -6 & -4 \end{pmatrix}.$$

Find all the eigenvalue of T and compute their respective eigenspaces.

***Solution**:* We look for values of λ for which the equation

$$T(v) = \lambda v$$

has nontrivial solutions. This is equivalent to finding values of λ for which the homogeneous system

$$(A - \lambda I)v = \mathbf{0} \tag{5}$$

has nontrivial solutions. The system in (5) has nontrivial solutions when the columns of the matrix

$$A - \lambda I = \begin{pmatrix} 5 - \lambda & 3 \\ -6 & -4 - \lambda \end{pmatrix}$$

are linearly dependent. This happens when $\det(A_\lambda I) = 0$, or

$$(\lambda - 5)(\lambda + 4) + 18 = 0. \tag{6}$$

Solving the equation (6) for λ yields the values $\lambda_1 = -1$ and $\lambda_2 = 2$. These are the eigenvalues of T.

To find $E_T(-1)$ we solve the homogeneous system

$$(A + I)v = \mathbf{0},$$

or

$$\begin{pmatrix} 6 & 3 \\ -6 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{7}$$

Using Gaussian elimination we see that the system in (7) is equivalent to the equation

$$x + \frac{1}{2}y = 0,$$

which has solution space given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

It then follows that

$$E_T(-1) = \text{span}\left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}.$$

Similar calculations show that

$$E_T(2) = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

□

4. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{for all} \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,$$

where A is the 2×2 matrix

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

where a and b are real constants.

(a) Show that T has real eigenvalues.

Solution: We look for values of λ for which the system

$$(A - \lambda I)v = \mathbf{0} \tag{8}$$

has nontrivial solutions. This happens when the columns of the matrix

$$A - \lambda I = \begin{pmatrix} a - \lambda & b \\ b & a - \lambda \end{pmatrix}$$

are linearly dependent. Thus, we require that $\det(A - \lambda I) = 0$, or

$$(\lambda - a)^2 - b^2 = 0. \tag{9}$$

We can factor the equation in (9) to get

$$(\lambda - a + b)(\lambda - a - b) = 0,$$

which yields the values $\lambda_1 = a - b$ and $\lambda_2 = a + b$, both of which are real. □

(b) Under what conditions on a and b will the eigenvalues obtained in part (a) be distinct eigenvalues?

Solution: $\lambda_1 = \lambda_2$ if and only if and only if $a - b = a + b$, which occurs if and only if $b = 0$. □

5. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Prove that $\lambda = 0$ is an eigenvalue of T if and only if T is not one–to–one.

Solution: $\lambda = 0$ is an eigenvalue of T if and only if the equation

$$T(v) = 0\ v,$$

or

$$T(v) = \mathbf{0},$$

has a nontrivial solution. Thus, $\lambda = 0$ is an eigenvalue of T if and only if T is not one–to–one. □