## Solutions to Assignment #6

1. Let $W$ denote the solution space of the equation

$$3x_1 + 8x_2 + 2x_3 - x_4 + x_5 = 0.$$

Find a linearly independent subset, $S$, of $\mathbb{R}^5$ such that $W = \text{span}(S)$.

***Solution**:* Solve for $x_1$ in terms of the other variables to get

$$x_1 = -\frac{8}{3}x_2 - \frac{2}{3}x_3 + \frac{1}{3}x_4 - \frac{1}{3}x_5.$$

Setting $x_2 = -3t$, $x_3 = -3s$, $x_4 = 3r$ and $x_5 = -3q$, where $t, s, r, q$ are arbitrary parameters, we get that

$$\begin{array}{rcl} x_1 & = & 8t + 2s + r + q \\ x_2 & = & -3t \\ x_3 & = & -3s \\ x_4 & = & 3r \\ x_5 & = & -3q \end{array}$$

so that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = t\begin{pmatrix} 8 \\ -3 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 2 \\ 0 \\ -3 \\ 0 \\ 0 \end{pmatrix} + r\begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \\ 0 \end{pmatrix} + q\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -3 \end{pmatrix}.$$

Hence, the solution space, $W$, is spanned by the set

$$S = \left\{ \begin{pmatrix} 8 \\ -3 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -3 \end{pmatrix} \right\}.$$

To see that $S$ is linearly independent, consider the vector equation

$$c_1\begin{pmatrix} 8 \\ -3 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 0 \\ -3 \\ 0 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \\ 0 \end{pmatrix} + c_4\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which yields the system

$$\left\{\begin{array}{rcl} 8c_1 + 2c_2 + c_3 + c_4 & = & 0 \\ -3c_1 & = & 0 \\ -3c_2 & = & 0 \\ 3c_3 & = & 0 \\ -3c_4 & = & 0. \end{array}\right.$$

From the last four equations we get that

$$c_1 = c_2 = c_3 = c_4 = 0.$$

Hence, $S$ is linearly independent and $W = \text{span}(S)$. $\square$

2. Let $W$ denote the solution space of the system

$$\left\{\begin{array}{rcl} x_1 - 2x_2 - x_3 & = & 0 \\ 2x_1 - 3x_2 + x_3 & = & 0. \end{array}\right.$$

Find a linearly independent subset, $S$, of $\mathbb{R}^3$ such that $W = \text{span}(S)$.

***Solution**:* Multiplying the first equation by $-2$, adding the scalar multiple to the second equation, and replacing the second equation by the result yields the system

$$\left\{\begin{array}{rcl} x_1 - 2x_2 - x_3 & = & 0 \\ x_2 + 3x_3 & = & 0. \end{array}\right. \tag{1}$$

Next, multiply the second equation in system (1) by 2, add the scalar multiple to the first equation and replace the first equation by the result to get

$$\left\{\begin{array}{rcl} x_1 \quad + \quad 5x_3 & = & 0 \\ x_2 + 3x_3 & = & 0. \end{array}\right. \tag{2}$$

The system in (2) can now be solved for the leading variable $c_1$ and $c_2$ to get

$$\left\{\begin{array}{rcl} x_1 & = & -5x_3 \\ x_2 & = & -3x_3. \end{array}\right. \tag{3}$$

Setting $x_3 = -t$, where $t$ is an arbitrary parameter, we obtain the solutions

$$\begin{array}{rcl} x_1 & = & 5t \\ x_2 & = & 3t \\ x_3 & = & -t. \end{array}$$

It then follows that the solution space of the system is

$$W = \text{ span}\left\{\begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix}\right\}.$$

Setting

$$S = \left\{\begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix}\right\}$$

completes the solution to the problem. □

3. In the following system, find the value or values of $\lambda$ for which the system has nontrivial solutions. In each case, give a a linearly independent subset of $\mathbb{R}^2$ which generates the solution space.

$$\left\{\begin{array}{rcl} (\lambda-3)x + \quad y & = & 0 \\ x + (\lambda-3)y & = & 0 \end{array}\right.$$

***Solution**:* Solve for $y$ in the first equation and substitute into the second to get

$$x - (\lambda-3)^2 x = 0,$$

which factors into

$$x[1-(\lambda-3)^2] = 0.$$

If $x = 0$, we get from the first equation that $y = 0$, and so we get the trivial solution. Hence, since we are looking for non–trivial solutions, we must have that

$$1 - (\lambda-3)^2 = 0.$$

This quadratic equation can be solve to yield

$$\lambda - 3 = \pm 1,$$

so that

$$\lambda = 2 \quad \text{ or } \quad \lambda = 4.$$

In the case that $\lambda = 2$ we get the system

$$\left\{\begin{array}{rcl} -x + y & = & 0 \\ x - y & = & 0 \end{array}\right.$$

which reduces to the equation

$$x = y.$$

Setting $y = t$, where $t$ is an arbitrary parameter, we get that

$$\begin{cases} x &= t \\ y &= t \end{cases}$$

so that, for $\lambda = 2$, the solution space is

$$\text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}.$$

On the other hand, if $\lambda = 4$, we obtain the system

$$\begin{cases} x + y &= 0 \\ x + y &= 0 \end{cases}$$

which reduces to the equation

$$x = -y.$$

Setting $y = -t$, where $t$ is an arbitrary parameter, we get that

$$\begin{cases} x &= \phantom{-}t \\ y &= -t \end{cases}$$

so that, for $\lambda = 2$, the solution space is

$$\text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}.$$

$\square$

4. Let $v \in \mathbb{R}^n$ and $S$ be a subset of $\mathbb{R}^n$.

   (a) Show that the set $\{v\}$ is linearly independent if and only if $v \neq \mathbf{0}$.

   *Proof:* Suppose first that $\{v\}$ is linearly independent. If $v = \mathbf{0}$, then

$$cv = \mathbf{0}$$

for any scalar $c$. It then follows that the equation

$$cv = \mathbf{0}$$

has nontrivial solutions and therefore $\{v\}$ is linearly dependent. But this contradicts the assumption of independence. We therefore conclude that $v \neq \mathbf{0}$.

Conversely, suppose that $v \neq \mathbf{0}$, and consider the equation

$$cv = \mathbf{0}.$$

Since $v \neq \mathbf{0}$, we must have that $c = 0$ and therefore

$$cv = \mathbf{0}$$

has only the trivial solution $c = 0$. Consequently, $\{v\}$ is linearly independent. $\square$

(b) Show that if $\mathbf{0} \in S$, then $S$ is linearly dependent.

*Proof:* If $S = \{\mathbf{0}\}$, then $S$ is linearly dependent by part (a). Thus, suppose that $S \neq \{\mathbf{0}\}$. Then, there exists $v \in S$ such that $v \neq \mathbf{0}$. Observe that

$$\mathbf{0} = 0 \cdot v$$

so that $\mathbf{0}$ is in the span of $v$, and therefore $S$ is linearly dependent. $\square$

5. Let $v_1$ and $v_2$ be vectors in $\mathbb{R}^n$, and let $c$ be a scalar.

(a) Show that $\{v_1, v_2\}$ is linearly independent if and only if $\{v_1, cv_1 + v_2\}$ is also linearly independent.

*Proof:* First observe that if $c = 0$, $\{v_1, v_2\}$ and $\{v_1, cv_1 + v_2\}$ are the same set. So the result holds true in this case. Thus, assume for the rest of the proof that $c \neq 0$.

Suppose that that $\{v_1, v_2\}$ is linearly independent and consider the equation

$$c_1 v_1 + c_2(cv_1 + v_2) = \mathbf{0}.$$

Using the distributive and associative properties we get that

$$(c_1 + cc_2)v_1 + cc_2 v_2 = \mathbf{0}.$$

It then follows from the linear independence of $\{v_1, v_2\}$ that

$$\begin{aligned} c_1 + cc_2 &= 0 \\ cc_2 &= 0. \end{aligned}$$

Since $c \neq 0$, we deduce from the above equations that $c_1 = c_2 = 0$ is the only solution of the system. Therefore, the set $\{v_1, cv_1 + v_2\}$ is linearly independent.

Conversely, suppose that $\{v_1, cv_1 + v_2\}$ is linearly independent and consider the vector equation

$$c_1 v_1 + c_2 v_2 = \mathbf{0}.$$

Adding $cc_2v_1 - cc_2v_1 = \mathbf{0}$ to both sides of the equation we get

$$c_1 v_1 + c_2 v_2 + cc_2 v_1 - cc_2 v_1 = \mathbf{0},$$

which by virtue of the distributive and associative properties can be written as

$$(c_1 - cc_2)v_1 + c_2(cv_1 + v_2) = \mathbf{0}.$$

Thus, since $\{v_1, cv_1 + v_2\}$ is linearly independent, it follows that

$$\begin{aligned} c_1 - cc_2 &= 0 \\ c_2 &= 0, \end{aligned}$$

from which we get that $c_1 = c_2 = 0$. Hence, $\{v_1, v_2\}$ is linearly independent.

$\square$

(b) Show that

$$\text{span}(\{v_1, v_2\}) = \text{span}(\{v_1, cv_1 + v_2\}).$$

*Proof:* Observe that $cv_1 + v_2 \in \text{span}(\{v_1, v_2\})$. Therefore,

$$\{v_1, cv_1 + v_2\} \subseteq \text{span}(\{v_1, v_2\}).$$

It then follows that

$$\text{span}\{v_1, cv_1 + v_2\} \subseteq \text{span}(\{v_1, v_2\}),$$

since $\text{span}\{v_1, cv_1 + v_2\}$ is the smallest subspace of $\mathbb{R}^n$ which contains $\{v_1, cv_1 + v_2\}$. Therefore, it suffices to show that

$$\text{span}(\{v_1, v_2\}) \subseteq \text{span}\{v_1, cv_1 + v_2\}.$$

To see why the last inclusion is true, observe that

$$v_2 = v_2 + cv_1 - cv_1 = -cv_1 + (cv_1 + v_2),$$

which is a linear combination of $v_1$ and $cv_1 + v_2$. It then follows that $v_2 \in \operatorname{span}\{v_1, cv_1 + v_2\}$ and therefore

$$\{v_1, v_2\} \subseteq \operatorname{span}\{v_1, cv_1 + v_2\}.$$

The last inclusion implies that

$$\operatorname{span}\{v_1, v_2\} \subseteq \operatorname{span}\{v_1, cv_1 + v_2\}$$

because $\operatorname{span}\{v_1, v_2\}$ is the smallest subspace of $\mathbb{R}^n$ which contains $\{v_1, v_2\}$.
$\square$