Solutions to Assignment #8

1. Given two subsets A and B of $\mathbb{R}^n$, the **union** of A and B, denoted by $A \cup B$, is the set which contains all vectors that are in either A or B; in symbols,

$$A \cup B = \{v \in \mathbb{R}^n \mid v \in A \text{ or } v \in B\}.$$

   (a) Prove that $A \subseteq A \cup B$ and $B \subseteq A \cup B$.

   Proof: Let $x \in A$; then, certainly, $x \in A$ or $x \in B$. Therefore,

$$x \in A \Rightarrow x \in A \text{ or } x \in B;$$

   or

$$x \in A \Rightarrow x \in A \cup B.$$

   Consequently, $A \subseteq A \cup B$.
   A similar argument shows that $B \subseteq A \cap B$. □

   (b) Suppose that W_1 and W_2 are two subspaces of $\mathbb{R}^2$. Give an example that shows that $W_1 \cup W_2$ is not necessarily a subspace of $\mathbb{R}^2$.

   Solution: Let $W_1 = \text{ span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$ and $W_2 = \text{ span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$.
   Then $W_1 \cup W_2$ contains all scalar multiples of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or all scalar multiples of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In particular, vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are in the union of W_1 and W_2; however, their sum

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

   is not in $W_1 \cup W_2$. Thus, $W_1 \cup W_2$ is not closed under vector addition. Hence, it is not a subspace of $\mathbb{R}^2$. □

2. Given two subsets A and B of $\mathbb{R}^n$, the **sum** of A and B, denoted by $A + B$, is the set which contains all vectors sums, $v + w$, such that $v \in A$ and $v \in B$; in symbols,

$$A + B = \{u \in \mathbb{R}^n \mid u = v + w, \text{ where } v \in A \text{ and } v \in B\}.$$

Prove that if W_1 and W_2 are two subspaces of $\mathbb{R}^n$, then $W_1 + W_2$ is also a subspace of $\mathbb{R}^n$.

Proof: Assume that W_1 and W_2 are subspaces of $\mathbb{R}^n$.

First, observe that, since W_1 and W_2 are subspaces of $\mathbb{R}^n$, then $0 \in W_1$ and $0 \in W_2$; so that $0 = 0 + 0 \in W_1 + W_2$, and therefore $W_1 + W_2$ is not empty.

Next, we show that $W_1 + W_2$ is closed under vector addition and scalar multiplication in $\mathbb{R}^n$.

Let $v \in W_1 + W_2$; then, $v = v_1 + v_2$, where $v_1 \in W_1$ and $v_2 \in W_2$. Then, for any scalar t, we get, by the distributive property,

$$tv = t(v_1 + tv_1) = tv_1 + tv_2,$$

where $tv_1 \in W_1$ and $tv_2 \in W_2$, since W_1 and W_2 are subspaces of $\mathbb{R}^n$. Consequently, $tv \in W_1 + W_2$ and so $W_1 + W_2$ is closed under scalar multiplication.

Next, let $v, w \in W_1 + W_2$. Then, $v = v_1 + v_2$ and $w = w_1 + w_2$, where $v_1, w_1 \in W_1$ and $v_2, w_2 \in W_2$. Then, by the associative and commutative properties of vector addition,

$$v + w = (v_1 + v_2) + (w_1 + w_2) = (v_1 + w_1) + (v_2 + w_2),$$

where $v_1 + w_1 \in W_1$ and $v_2 + w_2 \in W_2$, since W_1 and W_2 are subspaces of $\mathbb{R}^n$. Thus, $v + w \in W_1 + W_2$ and so $W_1 + W_2$ is closed under vector addition. □

3. Let W_1 and W_2 be two subspaces of $\mathbb{R}^n$ and define $W_1 + W_2$ as in the previous problem. Prove that $W_1 \cap W_2$, W_1 and W_2 are subspaces of $W_1 + W_2$.

Proof: First, observe that $W_1 \subseteq W_1 + W_2$. To see why this is so, let $v \in W_1$. Then,

$$v = v + \mathbf{0},$$

where $\mathbf{0} \in W_2$, since W_2 is a subspace. Consequently, $v \in W_1 + W_2$.

Similarly, we can prove that $W_2 \subseteq W_1 + W_2$.

Finally, since $W_1 \cap W_2$ is a subspace of both W_1 and W_2, by Problem 5(b) in Assignment #4, we obtain that $W_1 \cap W_2$, W_1 and W_2 are aall subspaces of $W_1 + W_2$. □

4. Let W_1 and W_2 be two subspaces of $\mathbb{R}^n$ and define $W_1 + W_2$ as in Problem 2 above. Suppose that $W_1 = \text{span}(S_1)$ and $W_2 = \text{span}(S_2)$, where $S_1 \subseteq W_1$ and $S_2 \subseteq W_2$. Prove that

$$W_1 + W_2 = \text{span}(S_1 \cup S_2).$$

Proof: Since $S_1 \subseteq S_1 \cup S_2$, it follows from Problem 1(a) in Assignment #5 that

$$\text{span}(S_1) \subseteq \text{span}(S_1 \cup S_2);$$

Thus,

$$W_1 \subseteq \text{span}(S_1 \cup S_2).$$

Similarly,

$$W_2 \subseteq \text{span}(S_1 \cup S_2).$$

Thus, since $\text{span}(S_1 \cup S_2)$ is a subspace of $\mathbb{R}^n$ and hence closed under vector addition, it follows that

$$v_1 + v_2 \in \text{span}(S_1 \cup S_2) \quad \text{for all } v_1 \in W_1 \text{ and } v_2 \in W_2;$$

that is,

$$W_1 + W_2 \subseteq \text{span}(S_1 \cup S_2). \tag{1}$$

To show the reverse inclusion, we first show that

$$S_1 \cup S_2 \subseteq W_1 + W_2. \tag{2}$$

To see why (2) is true, take $v \in S_1 \cup S_2$; then, either $v \in S_1$ or $v \in S_2$. If $v \in S_1$, then, given that $S_1 \subseteq W_1$,

$$v = v + 0 \in W_1 + W_2.$$

On the other hand, if $v \in S_2$, then, given that $S_2 \subseteq W_2$, we have that

$$v = 0 + v \in W_1 + W_2.$$

In either case we see that $v \in S_1 \cup S_2$ implies that $v \in W_1 + W_2$, which implies (2)

Now, It follows from (2) that

$$\text{span}(S_1 \cup S_2) \subseteq W_1 + W_2, \tag{3}$$

since span$(S_1 \cup S_2)$ is the smallest subspace of $\mathbb{R}^n$ which contains $S_1 \cup S_2$. Combining (1) and (3) yields the result. □

5. Let S_1 and S_2 be two linearly independent subsets of $\mathbb{R}^n$. When can we say that $S_1 \cup S_2$ is linearly independent? Explain your reasoning.

   Solution:
   Claim: Suppose that S_1 and S_2 are linearly independent. Then, $S_1 \cup S_2$ is linearly independent if $v \notin$ span$((S_1 \cup S_2)\backslash\{v\})$ for all $v \in S_1 \cup S_2$. This is, precisely, the definition of linear independence. □