## Assignment #9

**Due on Wednesday, October 8, 2014**

**Read** Section 2.8 on *Maximal Linearly Independent Subsets,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

**Read** Section 2.9 on *Bases,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

**Read** Section 1.6 on *Bases and Dimension* in Damiano and Little (pp. 47–55)

### Background and Definitions

- *(Definition of basis for a subspace of* $\mathbb{R}^n$*).* Let $W$ be a subspace of $\mathbb{R}^n$. A subset, $B$, of $W$ is said to be a **basis** for $W$ if and only if

  (i) $B$ is linearly independent, and

  (ii) $W = \text{span}(B)$.

- *(Column space of a matrix).* The **column space** of a matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \tag{1}$$

  denoted by $C_A$, is the span of the columns of $A$. That is,

$$C_A = \text{span}\left\{ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \ldots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \right\}.$$

  Thus, $C_A$ is a subspace of $\mathbb{R}^m$.

- *(Null space of a matrix).* The **null space** of the matrix $A$ defined in (1), denoted by $N_A$, is the solution space of the homogenous linear system

$$\left\{ \begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & 0 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & 0. \end{array} \right.$$

  Thus, $N_A$ is a subspace of $\mathbb{R}^n$.

**Do** the following problems

1. Let
$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; 2x + 3y - z = 0 \right\}.$$
Find a basis for $W$.

2. Let $A$ denote the matrix
$$\begin{pmatrix} 1 & 3 & -1 & 0 \\ 2 & 2 & 2 & 4 \\ 1 & 0 & 2 & 3 \end{pmatrix}. \qquad (2)$$
Find a basis for the column space, $C_A$, of the matrix $A$.

3. Find a basis for the null space, $N_A$, of the matrix, $A$, defined in (2).

4. Given a subset, $S$, or $\mathbb{R}^n$, and $v \in S$, the expression $S\backslash\{v\}$ denotes the set obtained by removing the vector $v$ from $S$.

   A subset, $S$, of a subspace, $W$, of $\mathbb{R}^n$ is said to be a **minimal generating set** for $W$ iff

   (i) $W = \text{span}(S)$, and

   (ii) for any $v$ in $S$, the set $S\backslash\{v\}$ does not span $W$.

   Prove that a minimal generating set for $W$ must be linearly independent.

   *Suggestion:* Argue by contradiction; that is, start out your argument assuming that $S$ is a minimal generating set for $W$, but $S$ is linearly dependent. Then, derive a contradiction.

5. Let $\{v_1, v_2, \ldots, v_n\}$ be a subset of $n$ vectors in $\mathbb{R}^n$. Prove that if $\{v_1, v_2, \ldots, v_n\}$ is linearly independent, then it must also span $\mathbb{R}^n$.