Definition of Real Vector Spaces or Linear Spaces

A **vector space**, or linear space, V, over the real numbers is a set of objects, called **vectors**, in which two algebraic operations, **vector addition** and **scalar multiplication**, have been defined. The first operation takes two members of V, call them u and v, and yields the *vector sum* of u and v, denoted $u+v$. The second operation combines a real number a, also known as a *scalar*, and a vector v in V to yield an object av called the *product* of a and v. The two operations must satisfy the following properties:

I. Closure properties

0. If u and v are vectors in V, then $u+v$ is also a vector in V. If v is in V and a is a real number, then the product av is also in V.

II. Properties of vector addition

1. For any u and v in V, $u+v=v+u$ (*commutativity of vector addition*).
2. For any three elements u, v, and w in V, $(u+v)+w=u+(v+w)$ (*associativity of vector addition*).
3. There exists an element $\mathbf{0}$ in V, called the *zero vector,* with the property: $v+0=v$ for all v in V (*existence of an identity for vector addition*).
4. For every v in V, there exists a u, also in V, with the property: $u+v=0$ (*existence of additive inverses*).

III. Properties of scalar multiplication

5. For any pair of real numbers a and b, and any vector v in V, $(ab)v=a(bv)$.
6. For any v in V, $1v=v$.

IV. Distributive properties

7. For any scalar a and any pair of vectors u and v, $a(u+v)=au+av$.
8. For any scalars a and b, and any vector v, $(a+b)v=av+bv$.