Assignment #10

Due on Tuesday, November 29, 2016

Read Section 8.1 on the *Definition of Convergence in Distribution* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 8.2 on the *mgf Convergence Theorem* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 8.3 on the *Central Limit Theorem* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 5.4 on *The Poisson Distribution* in DeGroot and Schervish.

Read Section 5.6 on *The Normal Distribution* in DeGroot and Schervish.

Read Section 6.3 on *The Central Limit Theorem* in DeGroot and Schervish.

Background and Definitions

Definition (Convergence in Distribution)**.** Let (X_n) be a sequence of random variables with cumulative distribution functions F_{X_n}, for $n = 1, 2, 3, \ldots$, and Y be a random variable with cdf F_Y. We say that the sequence (X_n) converges to Y in distribution, if

$$\lim_{n\to\infty} F_{X_n}(x) = F_Y(x)$$

for all x where F_Y is continuous. The distribution of Y is usually called the **limiting distribution** of the sequence (X_n).

Theorem (mgf Convergence Theorem)**.** *Let (X_n) be a sequence of random variables with moment generating functions $\psi_{X_n}(t)$, for $|t| < h$, $n = 1, 2, 3, \ldots$, and some positive number h. Suppose Y has mgf $\psi_Y(t)$ which exists for $|t| < h$. Then, if*

$$\lim_{n\to\infty} \psi_{X_n}(t) = \psi_Y(t), \quad \textit{for } |t| < h,$$

it follows that $\lim\limits_{n\to\infty} F_{X_n}(x) = F_Y(x)$ for all x where F_Y is continuous.

Do the following problems.

1. Let (X_k) denote a sequence of independent identically distributed random variables such that $X_k \sim \text{Normal}(\mu, \sigma^2)$ for every $k = 1, 2, \ldots$, and for some $\mu \in \mathbb{R}$ and $\sigma > 0$.

(a) For each $n \geqslant 1$, define $\overline{X}_n = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$.

Determine the mgf, $\psi_{\overline{X}_n}(t)$, for $\overline{X}_n$, and compute $\lim\limits_{n\to\infty} \psi_{\overline{X}_n}(t)$.

Give the limiting distribution of $\overline{X}_n$ as $n \to \infty$.

(b) Define $Z_n = \dfrac{\overline{X}_n - \mu}{\sigma/\sqrt{n}}$ for all $n \geqslant 1$.

Determine the mgf, $\psi_{Z_n}(t)$, for Z_n, and compute $\lim\limits_{n\to\infty} \psi_{Z_n}(t)$.

Find the limiting distribution of Z_n as $n \to \infty$.

2. Let $q = 0.95$ denote the probability that a person, in certain age group, lives at least 5 years.

   (a) If we observe 60 people from that group and assume independence, what is the probability that at least 56 of them live 5 years or more?

   (b) Find and approximation to the result of part (a) using the Poisson distribution.

3. Let $Y_n \sim \text{Binomial}(n, p)$, for $n = 1, 2, 3, \ldots$, and define $Z_n = \dfrac{Y_n - np}{\sqrt{np(1-p)}}$ for $n = 1, 2, 3, \ldots$ Use the Central Limit Theorem to find the limiting distribution of Z_n.

   Suggestion: Recall that Y_n is the sum of n independent Bernoulli(p) trials.

4. Suppose that 75% of the people in a certain metropolitan area live in the city and 25% of the people live in the suburbs. If 1200 people attending certain concert represent a random sample from the metropolitan area, what is the probability that the number of people from the suburbs attending the concert will be fewer than 270? State any assumption that make in your solution to this problem.

5. Suppose that a random sample of size n is to be taken from a distribution for which the mean is μ and the standard deviation is 3. Use the Central Limit Theorem to determine approximately the smallest value of n for which the following relation will be satisfied: $\Pr(|\overline{X}_n - \mu| < 0.3) \geqslant 0.95$.

6. An experiment consists of rolling a die 81 times and computing the average of the numbers on the top face of the die. Estimate the probability that the sample mean will be less than 3.

7. A random sample of size 49 is taken form a distribution with mean μ and variance σ^2. Estimate the probability that sample mean will be within 0.7 standard deviations from the mean of the distribution.

8. A large freight elevator can transport a maximum of 9800 pounds. Suppose a load of cargo containing 49 boxes must be transported via the elevator. Experience has shown that the weight of boxes of this type of cargo follows a distribution with mean $\mu = 205$ pounds and standard deviation $\sigma = 15$ pounds. Based on this information, what is the probability that all 49 boxes can be safely loaded onto the freight elevator and transported?

9. Forty–nine measurements are recorded to several decimal places. Each of these 49 numbers is rounded off to the nearest integer. The sum of the original 49 numbers is approximated by the sum of those integers. Assume that the errors made in rounding off are independent, identically distributed random variables with a uniform distribution over the interval $(-0.5, 0.5)$. Compute approximately the probability that the sum of the integers is within two units of the true sum.

10. Let X denote a random variable with pdf

$$f_X(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } 1 < x < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Consider a random sample of size 72 from this distribution. Compute approximately the probability that 50 or more observations of the random sample are less than 3.