Assignment #2

Due on Thursday, September 15, 2016

Read Section 3.4 on *Defining a Probability Function* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Read Section 1.5 on *The Definition of Probability* in DeGroot and Schervish.

Read Section 1.6 on *Finite Sample Spaces* in DeGroot and Schervish.

Do the following problems.

1. Consider two events A and B such that $\Pr(A) = 1/3$ and $\Pr(B) = 1/2$. Determine the value of $\Pr(B \cap A^c)$ for each of the following conditions:

   (a) A and B are disjoint;

   (b) $A \subseteq B$;

   (c) $\Pr(A \cap B) = 1/8$.

2. Consider two events A and B with $\Pr(A) = 0.4$ and $\Pr(B) = 0.7$. Determine the maximum and minimum possible values for $\Pr(A \cap B)$ and the conditions under which each of these values is attained.

3. Prove that for every two events A and B, the probability that exactly one of the two events will occur is given by the expression

$$\Pr(A) + \Pr(B) - 2\Pr(A \cap B).$$

4. Let A and B be elements in a σ–field $\mathcal{B}$ on a sample space $\mathcal{C}$, and let Pr denote a probability function defined on $\mathcal{B}$. Recall that $A \setminus B = \{x \in A \mid x \notin B\}$. Prove that if $B \subseteq A$, then

$$\Pr(A \setminus B) = \Pr(A) - \Pr(B).$$

5. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ denote a probability space, and B an event in $\mathcal{B}$ with $\Pr(B) > 0$. Let
$$\mathcal{B}_B = \{D \subset \mathcal{C} \mid D = E \cap B \text{ for some } E \in \mathcal{B}\}.$$
We have already seen that $\mathcal{B}_B$ is a σ–field.

   Let $P_B \colon \mathcal{B}_B \to \mathbb{R}$ be defined by $P_B(A) = \dfrac{\Pr(A)}{\Pr(B)}$ for all $A \in \mathcal{B}_B$. Verify that $(B, \mathcal{B}_B, P_B)$ is a probability space; that is, show that $P_B \colon \mathcal{B}_B \to \mathbb{R}$ is a probability function.

6. Let $(\mathcal{C}, \mathcal{B}, \Pr)$ be a sample space. Suppose that $E_1, E_2, E_3, \ldots$ is a sequence of events in $\mathcal{B}$ satisfying
$$E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots .$$
Prove that $\lim\limits_{n\to\infty} \Pr(E_n) = \Pr\left(\bigcap_{k=1}^{\infty} E_k\right)$.

   Hint: Use the analogous result for an increasing nested sequence of events presented in class and De Morgan's laws.

7. A point (x, y) is to be selected at random from a square S containing all the points (x, y) such that $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1$. Suppose that the probability that the selected point will belong to each specified subset of S is equal to the area of that subset. Find the probability of each of the following subsets:

   (a) the subset of points such that $\left(x - \dfrac{1}{2}\right)^2 + \left(y - \dfrac{1}{2}\right)^2 \geqslant \dfrac{1}{4}$;

   (b) the subset of points such that $\dfrac{1}{2} < x + y < \dfrac{3}{2}$;

   (c) the subset of points such that $y < 1 - x^2$;

   (d) the subset of points such that $x = y$.

8. In a random experiment, two balanced dice are rolled.

   (a) What is the probability that the sum of the two numbers that appear will be even?

   (b) What is the probability that the difference of the two numbers that appear will be less than 3?

9. A coin is tossed as many times as necessary to turn up one head. Thus, the elements of the sample space $\mathcal{C}$ corresponding to this experiment are

$$H, TH, TTH, TTTH, \ldots$$

Let Pr be a functions that assigns to these elements the values $\dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \dfrac{1}{16}, \ldots$ respectively.

   (a) Show that $\Pr(\mathcal{C}) = 1$.

   (b) Let E_1 denote the event $E_1 = \{H, TH, TTH, TTTH \text{ or } TTTTH\}$, and compute $\Pr(E_1)$.

   (c) Let $E_2 = \{TTTTH, TTTTTH\}$, and compute $\Pr(E_2)$, $\Pr(E_1 \cap E_2)$ and $\Pr(E_2 \setminus E_1)$

10. Let $\mathcal{C} = \{x \in \mathbb{R} \mid x > 0\}$ and define Pr on open intervals (a, b) with $0 < a < b$ by

$$\Pr((a, b)) = \int_a^b e^{-x} \ \mathrm{d}x.$$

   (a) Show that $\Pr(\mathcal{C}) = 1$.

   (b) Let $E = \{x \in \mathcal{C} \mid 4 < x < \infty\}$, and compute $\Pr(E)$, $\Pr(E^c)$ and $\Pr(E \cup E^c)$.