Solutions to Review Problems for Exam 2

1. A bowl contains 5 chips of the same size and shape. Two chips are red and the other three are blue. Draw three chips from the bowl at random, without replacement. Let X denote the number of blue chips in a drawing.

   (a) Give the pmf of X.

   ***Solution**:* Possible values of X are 1, 2 and 3.

   Compute, using equal likelihood assumption and the fact that the sampling is done without replacement,

$$\Pr(X = 1) = \frac{\binom{3}{1} \cdot \binom{2}{2}}{\binom{5}{3}} = \frac{3}{10}.$$

   Similarly

$$\Pr(X = 2) = \frac{\binom{3}{2} \cdot \binom{2}{1}}{\binom{5}{3}} = \frac{3}{5},$$

   and

$$\Pr(X = 3) = \frac{\binom{3}{3} \cdot \binom{2}{0}}{\binom{5}{3}} = \frac{1}{10}.$$

   We then have that the pmf of X is

$$p_X(k) = \begin{cases} \dfrac{3}{10}, & \text{if } k = 1; \\[2ex] \dfrac{3}{5}, & \text{if } k = 2; \\[2ex] \dfrac{1}{10}, & \text{if } k = 3; \\[2ex] 0, & \text{elsewhere.} \end{cases} \tag{1}$$

   □

(b) Compute $\Pr(X > 1)$.

Solution: Use the definition of the pmf of X in (1) to get

$$\Pr(X > 1) = 1 - \Pr(X \leqslant 1) = 1 - p_X(1) = \frac{7}{10},$$

or 70%. □

(c) Compute $E(X)$.

Solution: Using the definition of the pmf of X in (1), we compute

$$\begin{aligned} E(X) &= \sum_{k=1}^{3} k p_X(k) \\ &= 1 \cdot \frac{3}{10} + 2 \cdot \frac{3}{5} + 3 \cdot \frac{1}{10} \\ &= 1 \cdot \frac{18}{10}, \end{aligned}$$

or $E(X) = 1.8$. □

2. Let X have pmf given by $p_X(x) = \dfrac{1}{3}$ for $x = 1, 2, 3$ and $p(x) = 0$ elsewhere. Give the pmf of $Y = 2X + 1$.

Solution: Note that the possible values for Y are 3, 5 and 7

Compute

$$\Pr(Y = 3) = \Pr(2X + 1 = 3) = \Pr(X = 1) = \frac{1}{3}.$$

Similarly, we get that

$$\Pr(Y = 5) = \Pr(X = 2) = \frac{1}{3},$$

and

$$\Pr(Y = 7) = \Pr(X = 3) = \frac{1}{3}.$$

Thus,

$$p_Y(k) = \begin{cases} \dfrac{1}{3} & \text{for } k = 3, 5, 7; \\ \\ 0 & \text{elsewhere.} \end{cases}$$

□

3. Let

$$f_X(x) = \begin{cases} \dfrac{1}{x^2}, & \text{if } 1 < x < \infty; \\ \\ 0, & \text{if } x \leqslant 1, \end{cases}$$

be the pdf of a random variable X. If E_1 denote the interval $(1, 2)$ and E_2 the interval $(4, 5)$, compute $\Pr(E_1)$, $\Pr(E_2)$, $\Pr(E_1 \cup E_2)$ and $\Pr(E_1 \cap E_2)$.

***Solution**:* Compute

$$\Pr(E_1) = \Pr(1 < X < 2) = \int_1^2 \frac{1}{x^2}\, dx = -\frac{1}{x}\bigg|_1^2 = \frac{1}{2},$$

$$\Pr(E_2) = \Pr(4 < X < 5) = \int_4^5 \frac{1}{x^2}\, dx = -\frac{1}{x}\bigg|_4^5 = \frac{1}{20},$$

$$\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2) = \frac{11}{20},$$

since E_1 and E_2 are mutually exclusive, and

$$\Pr(E_1 \cap E_2) = 0,$$

since E_1 and E_2 are mutually exclusive. □

4. A *mode* of a distribution of a random variable X is a value of x that maximizes the pdf or the pmf. If there is only one such value, it is called *the mode of the distribution.* Find the mode for each of the following distributions:

   (a) $p(x) = \left(\dfrac{1}{2}\right)^x$ for $x = 1, 2, 3, \ldots$, and $p(x) = 0$ elsewhere.

   ***Solution**:* Note that $p(x)$ is decreasing; so, $p(x)$ is maximized when $x = 1$. Thus, 1 is the mode of the distribution of X. □

   (b) $f(x) = \begin{cases} 12x^2(1-x), & \text{if } 0 < x < 1; \\ \\ 0 & \text{elsewhere.} \end{cases}$

   ***Solution**:* Maximize the function f over $[0, 1]$.
   Compute
   $$f'(x) = 24x(1-x) - 12x^2 = 12x(2-3x),$$
   so that f has a critical points at $x = 0$ and $x = \dfrac{2}{3}$.

Since $f(0) = f(1) = 0$ and $f(2/3) > 0$, it follows that f takes on its maximum value on $[0,1]$ at $x = \dfrac{2}{3}$. Thus, the mode of the distribution of X is $x = \dfrac{2}{3}$. □

5. Let X have pdf

$$f_X(x) = \begin{cases} 2x, & \text{if } 0 < x < 1; \\ 0, & \text{elsewhere.} \end{cases}$$

Compute the probability that X is at least 3/4, given that X is at least 1/2.

***Solution**:* We are asked to compute

$$\Pr(X \geqslant 3/4 \mid X \geqslant 1/2) = \frac{\Pr[(X \geqslant 3/4) \cap (X \geqslant 1/2)]}{\Pr(X \geqslant 1/2)}, \tag{2}$$

where

$$\begin{aligned} \Pr(X \geqslant 1/2) &= \int_{1/2}^{1} 2x\, dx \\ &= x^2 \Big|_{1/2}^{1} \\ &= 1 - \frac{1}{4}, \end{aligned}$$

so that

$$\Pr(X \geqslant 1/2) = \frac{3}{4}; \tag{3}$$

and

$$\begin{aligned} \Pr[(X \geqslant 3/4) \cap (X \geqslant 1/2)] &= \Pr(X \geqslant 3/4) \\ &= \int_{3/4}^{1} 2x\, dx \\ &= x^2 \Big|_{3/4}^{1} \\ &= 1 - \frac{9}{16}, \end{aligned}$$

so that

$$\Pr[(X \geqslant 3/4) \cap (X \geqslant 1/2)] = \frac{7}{16}. \tag{4}$$

Substituting (4) and (3) into (2) then yields

$$\Pr(X \geqslant 3/4 \mid X \geqslant 1/2) = \frac{\frac{7}{16}}{\frac{3}{4}} = \frac{7}{12}.$$

$\square$

6. Divide a segment at random into two parts. Find the probability that the largest segment is at least three times the shorter.

***Solution**:* Assume the segment is the interval $(0,1)$ and let $X \sim \text{Uniform}(0,1)$. Then X models a random point in $(0,1)$. We have two possibilities: Either $X \leqslant 1 - X$ or $X > 1 - X$; or, equivalently, $X \leqslant \frac{1}{2}$ or $X > \frac{1}{2}$.

Define the events

$$E_1 = \left(X \leqslant \frac{1}{2}\right) \quad \text{and } E_2 = \left(X > \frac{1}{2}\right).$$

Observe that $\Pr(E_1) = \frac{1}{2}$ and $\Pr(E_2) = \frac{1}{2}$.

The probability that the largest segment is at least three times the shorter is given by

$$\Pr(E_1)\Pr(1 - X > 3X \mid E_1) + \Pr(E_2)\Pr(X > 3(1 - X) \mid E_2),$$

by the Law of Total Probability, where

$$\Pr(1 - X > 3X \mid E_1) = \frac{\Pr[(X < 1/4) \cap E_1]}{\Pr(E_1)} = \frac{1/4}{1/2} = \frac{1}{2}.$$

Similarly,

$$\Pr(X > 3(1 - X) \mid E_2) = \frac{\Pr[(X > 3/4) \cap E_1]}{\Pr(E_2)} = \frac{1/4}{1/2} = \frac{1}{2}.$$

Thus, the probability that the largest segment is at least three times the shorter is

$$\Pr(E_1)\Pr(1 - X > 3X \mid E_1) + \Pr(E_2)\Pr(X > 3(1 - X) \mid E_2) = \frac{1}{2}.$$

$\square$

7. Let X have pdf

$$f_X(x) = \begin{cases} x^2/9, & \text{if } 0 < x < 3; \\ 0, & \text{elsewhere.} \end{cases}$$

Find the pdf of $Y = X^3$.

Solution: First, compute the cdf of Y

$$F_Y(y) = \Pr(Y \leqslant y). \tag{5}$$

Observe that, since $Y = X^3$ and the possible values of X range from 0 to 3, the values of Y will range from 0 to 27. Thus, in the calculations that follow, we will assume that $0 < y < 27$.

From (5) we get that

$$\begin{aligned} F_Y(y) &= \Pr(X^3 \leqslant y) \\ &= \Pr(X \leqslant y^{1/3}) \\ &= F_X(y^{1/3}) \end{aligned}$$

Thus, for $0 < y < 27$, we have that

$$f_Y(y) = f_X(y^{1/3}) \cdot \frac{1}{3} y^{-3/2}, \tag{6}$$

where we have applied the Chain Rule.

It follows from (6) and the definition of f_X that

$$f_Y(y) = \frac{1}{9}\left[y^{1/3}\right]^2 \cdot \frac{1}{3} y^{-3/2} = \frac{1}{27}, \quad \text{for } 0 < y < 27. \tag{7}$$

Combining (7) and the definition of f_X we obtain the pdf for Y:

$$f_Y(y) = \begin{cases} \dfrac{1}{27}, & \text{for } 0 < y < 27; \\ \\ 0 & \text{elsewhere;} \end{cases}$$

in other words, $Y \sim \text{Uniform}(0, 27)$. □

8. Assume that the random variable X has mgf
$$\psi_X(t) = \frac{e^t}{4-3e^t}, \qquad \text{for } t < \ln\left(\frac{4}{3}\right). \tag{8}$$
Compute the expected value, second moment and variance of X.

Solution: Write the mgf of X in (8) as
$$\psi_X(t) = (4e^{-t}-3)^{-1}, \qquad \text{for } t < \ln\left(\frac{4}{3}\right),$$
and differentiate with respect to t to get
$$\psi'_X(t) = (-1)(4e^{-t}-3)^{-2}\cdot(-4e^{-t}), \qquad \text{for } t < \ln\left(\frac{4}{3}\right),$$
where we have used the Chain Rule, or
$$\psi'_X(t) = 4e^{-t}(4e^{-t}-3)^{-2}, \qquad \text{for } t < \ln\left(\frac{4}{3}\right), \tag{9}$$
and, using the product rule,
$$\psi''_X(t) = -4e^{-t}(4e^{-t}-3)^{-2} - 2(4e^{t})(4e^{-t}-3)^{-3}\cdot(-4e^{-t}), \qquad \text{for } t < \ln\left(\frac{4}{3}\right),$$
which simplifies to
$$\begin{aligned} \psi''_X(t) &= -4e^{-t}(4e^{-t}-3)^{-2} - 2(4e^{t})4e^{-t}-3)^{-3}\cdot(-4e^{-t}) \\ &= 2(4e^{-t})^2(4e^{-t}-3)^{-3} - 4e^{-t}(4e^{-t}-3)^{-2} \\ &= 4e^{-t}(4e^{-t}-3)^{-3}(8e^{-t}-(4e^{-t}-3)), \end{aligned}$$
or
$$\psi''_X(t) = 4e^{-t}(4e^{-t}-3)^{-3}(4e^{-t}+3), \qquad \text{for } t < \ln\left(\frac{4}{3}\right). \tag{10}$$
Using (9) and (10) we then compute
$$E(X) = \psi'_X(0) = 4,$$
$$E(X^2) = \psi''_X(0) = 28,$$
and
$$\text{Var}(X) = E(X^2) - (E(X))^2 = 28 - 16 = 12.$$
□

9. Let X have mgf given by

$$\psi_X(t) = \frac{1}{3}e^t + \frac{2}{3}e^{2t}, \qquad \text{for } t \in \mathbb{R}. \tag{11}$$

(a) Give the distribution of X

***Solution**:* The mgf in (11) corresponds to a discrete random variable with pmf

$$p_X(k) = \begin{cases} \frac{1}{3}, & \text{if } k = 1; \\ \frac{2}{3}, & \text{if } k = 2; \\ 0, & \text{elsewhere.} \end{cases}$$

□

(b) Compute the expected value and variance of X.

***Solution**:* Compute the derivatives of the mgf in (11) to get

$$\psi_X'(t) = \frac{1}{3}e^t + \frac{4}{3}e^{2t}, \qquad \text{for } t \in \mathbb{R}, \tag{12}$$

and

$$\psi_X''(t) = \frac{1}{3}e^t + \frac{8}{3}e^{2t}, \qquad \text{for } t \in \mathbb{R}. \tag{13}$$

Using (12) and (13) we then obtain

$$E(X) = \psi_X'(0) = \frac{5}{3},$$

$$E(X^2) = \psi_X''(0) = 3.$$

Thus, the variance of X is

$$\text{Var}(X) = E(X^2) - (E(X))^2 = 3 - \frac{25}{9} = \frac{2}{9}.$$

□

10. Let X have mgf given by

$$f_X(x) = \begin{cases} \dfrac{e^t - e^{-t}}{2t}, & \text{if } t \neq 0; \\ \\ 1, & \text{if } t = 0. \end{cases} \tag{14}$$

   (a) Give the distribution of X

   Solution: Looking at the handout on special distributions we see that the mgf given in (14) corresponds to that of a Uniform$(-1, 1)$ random variable. Thus, by the mgf Uniqueness Theorem, $X \sim$ Uniform$(-1, 1)$, Consequently, the pdf of X is given by

$$f_X(x) = \begin{cases} \dfrac{1}{2}, & \text{if } -1 < x < 1; \\ \\ 0, & \text{elsewhere.} \end{cases}$$

   □

   (b) Compute the expected value and variance of X.

   Solution: The expected value and variance of X can also be obtained by reading the Special Distributions handout:

$$E(X) = \frac{-1+1}{2} = 0$$

   and

$$\text{Var}(X) = \frac{(1-(-1))^2}{12} = \frac{4}{12} = \frac{1}{3}.$$

   □