Assignment #3

Due on Wednesday, September 14, 2016

Section 3.1 on *Modeling Traffic Flow* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`.

Background and Definitions

Let $f\colon I \to \mathbf{R}$ denote a continuous, real–valued function defined on an open interval, I, of the real line. In this problem set you will establish the following result:

Proposition A. Suppose that $f\colon I \to \mathbf{R}$ is continuous and

$$\int_a^b f(x)\ dx = 0 \qquad \text{for all intervals } [a,b] \subset I; \tag{1}$$

then, $f(x) = 0$ for all $x \in I$.

Do the following problems.

1. Assume that $f\colon I \to \mathbf{R}$ is continuous and that $f(x_o) \neq 0$ for some $x_o \in I$. Use the definition of continuity at x_o, with $\varepsilon = \dfrac{|f(x_o)|}{2}$, to deduce that there exists $\delta > 0$ such that $[x_o - \delta, x_o + \delta] \subset I$ and

$$x \in [x_o - \delta, x_o + \delta] \Rightarrow f(x_o) - \frac{|f(x_o)|}{2} < f(x) < f(x_o) + \frac{|f(x_o)|}{2}. \tag{2}$$

2. Let f, x_o and δ be as in Problem 1. Use (2) to show that, if $f(x_o) > 0$, then

$$x \in [x_o - \delta, x_o + \delta] \Rightarrow f(x) > \frac{|f(x_o)|}{2}. \tag{3}$$

3. Let f, x_o and δ be as in Problem 1. Use (2) to show that, if $f(x_o) < 0$, then

$$x \in [x_o - \delta, x_o + \delta] \Rightarrow f(x) < -\frac{|f(x_o)|}{2}. \tag{4}$$

4. Let f, x_o and δ be as in Problem 1. Use the results in the previous problems in (3) and (4) to show that, if $f(x_o) \neq 0$, then either

$$\int_{x_o-\delta}^{x_o+\delta} f(x)\ dx > \delta|f(x_o)| > 0 \quad \text{or} \quad \int_{x_o-\delta}^{x_o+\delta} f(x)\ dx < -\delta|f(x_o)| < 0.$$

5. Prove Proposition A through an indirect argument; that is, assume that (1) holds true, but $f(x_o) \neq 0$ for some $x_o \in I$, and derive a contradiction.