

Assignment #9

Due on Friday, November 4, 2016

Read Section 4.2 on *Random Processes* in the class lecture notes at <http://pages.pomona.edu/~ajr04747/>.

Read Chapter 5 on *Diffusion* in the class lecture notes at <http://pages.pomona.edu/~ajr04747/>.

Background and Definitions The probability density function, $p(x, t)$, for the location, X_t , of a Brownian particle at time t ,

$$p(x, t) = \frac{1}{\sqrt{4\pi Dt}} \cdot e^{-x^2/(4Dt)}, \quad \text{for } x \in \mathbb{R} \text{ and } t > 0, \quad (1)$$

which was derived in class, is also called the heat kernel. In this set of problems, we derive a few properties of this function.

Do the following problems.

1. Verify that p solves the diffusion equation

$$\frac{\partial p}{\partial t} = D \frac{\partial^2 p}{\partial x^2}, \quad \text{for } x \in \mathbb{R} \text{ and } t > 0. \quad (2)$$

2. Explain why $\int_{-\infty}^{\infty} p(x, t) dx = 1$, for all $t > 0$. Give an interpretation to this result.

3. Show the following:

(a) If $x \neq 0$, then $\lim_{t \rightarrow 0^+} p(x, t) = 0$.

(b) If $x = 0$, then $\lim_{t \rightarrow 0^+} p(x, t) = +\infty$.

4. Use a mathematical software package to sketch the graph of $x \mapsto p(x, t)$ for several values of $t > 0$.
5. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a bounded function that is Riemann integrable on bounded intervals and define

$$u(x, t) = \int_{-\infty}^{\infty} p(x - y, t) f(y) dy, \quad \text{for } x \in \mathbb{R} \text{ and } t > 0.$$

Explain why u solves the diffusion equation

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}, \quad \text{for } x \in \mathbb{R} \text{ and } t > 0.$$