Review Problems for Exam 1

1. **Modeling the Spread of a Disease.** In a simple model for a disease that is spread through infections transmitted between individuals in a population, the population is divided into three compartments pictured in Figure 1. The

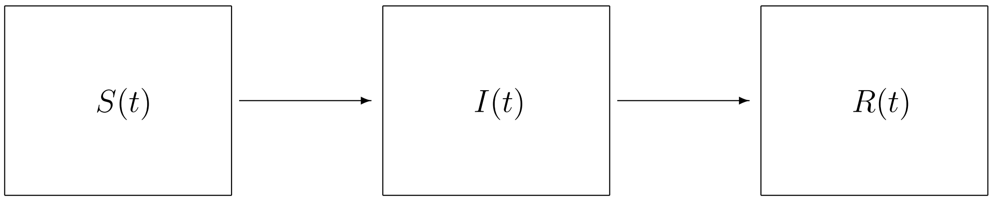

Figure 1: SIR Compartments

first compartment, $S(t)$, denotes the set of individuals in a population that are susceptible to acquiring the disease; the second compartment, $I(t)$, denotes the set of infected individual who can also infect others; and the third compartment, $R(t)$, denotes the set of individuals who had the disease and who have recovered from it; they can no longer get infected.

Assume that the total number of individuals in the population,

$$N = S(t) + I(t) + R(t),$$

is constant. Susceptible individuals can get infected by contact with infectious individuals and move to the infected class. This is indicated by the arrow going from the $S(t)$ compartment to the $I(t)$ compartment.

The rate at which susceptible individuals get infected is proportional to product of number of susceptible individuals and the number of infected individuals with constant of proportionality $\beta > 0$. The rate at which infected individuals recover is proportional to the number of infected individuals with constant of proportionality $\gamma > 0$. What are the units for β and γ?

Use conservation principles to derive a system of differential equations for the functions S, I and R, assuming that they are differentiable. Models of this type were first studied by Kermack and McKendrick in the early 1930s.

Introduce dimensionless variables

$$\widehat{s}(t) = \frac{S(t)}{N}, \quad \widehat{i}(t) = \frac{I(t)}{N}, \quad \widehat{r}(t) = \frac{R(t)}{N}, \quad \text{and} \quad \widehat{t} = \frac{t}{\tau},$$

for some scaling factor, τ, in units of time, in order to write the system in dimensionless form.

2. **Modeling Traffic Flow.** Consider the initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + g'(u)\dfrac{\partial u}{\partial x} &= 0; \\ \\ u(x,0) &= f(x), \end{cases}$$

where $g(u) = u(1-u)$, and the initial condition f is given by

$$f(x) = \begin{cases} 1, & \text{if } x < -1; \\ \dfrac{1}{2}(1-x), & \text{if } -1 \leqslant x < 1; \\ 0, & \text{if } x \geqslant 1. \end{cases}$$

(a) Sketch the characteristic curves of the partial differential equation.

(b) Explain how the initial value problem can be solved in this case, and give a formula for $u(x,t)$.

3. **Traffic Flow at a Red Light**. Let the initial condition in Problem 3 be given by $f(x) = 1$ for $x \leqslant 0$ and $f(x) = 0$ for $x > 0$.

(a) Explain why this initial value problem models the situation at a traffic light before the light turns green.

(b) Sketch the characteristic curves of the partial differential equation.

(c) Explain why a shock wave solution doe not develop at $t = 0$.

(d) Look for a solution to the equation of the form

$$u(x,t) = \varphi\left(\frac{x}{t}\right), \quad \text{for } -t < x < t, \quad \text{and} \quad t > 0,$$

where φ is a differentiable function of a single variable.

Suggestion: Introduce a new variable $\eta = \dfrac{x}{t}$, and compute $\dfrac{d\varphi}{d\eta}$.