## Solutions to Review Problems for Exam #1

1. **Modeling the Spread of a Disease.** In a simple model for a disease that is spread through infections transmitted between individuals in a population, the population is divided into three compartments pictured in Figure 1. The

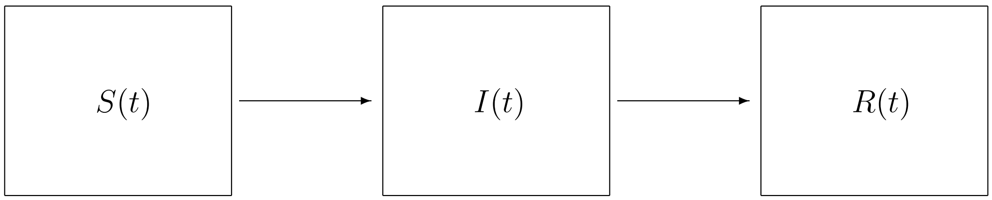


Figure 1: SIR Compartments

first compartment, $S(t)$, denotes the set of individuals in a population that are susceptible to acquiring the disease; the second compartment, $I(t)$, denotes the set of infected individual who can also infect others; and the third compartment, $R(t)$, denotes the set of individuals who had the disease and who have recovered from it; they can no longer get infected.

Assume that the total number of individuals in the population,

$$N = S(t) + I(t) + R(t),$$

is constant. Susceptible individuals can get infected by contact with infectious individuals and move to the infected class. This is indicated by the arrow going from the $S(t)$ compartment to the $I(t)$ compartment.

The rate at which susceptible individuals get infected is proportional to product of number of susceptible individuals and the number of infected individuals with constant of proportionality $\beta > 0$. The rate at which infected individuals recover is proportional to the number of infected individuals with constant of proportionality $\gamma > 0$. What are the units for $\beta$ and $\gamma$?

Use conservation principles to derive a system of differential equations for the functions $S$, $I$ and $R$, assuming that they are differentiable. Models of this type were first studied by Kermack and McKendrick in the early 1930s.

Introduce dimensionless variables

$$\widehat{s}(t) = \frac{S(t)}{N}, \quad \widehat{i}(t) = \frac{I(t)}{N}, \quad \widehat{r}(t) = \frac{R(t)}{N}, \quad \text{and} \quad \widehat{t} = \frac{t}{\tau}, \qquad (1)$$

for some scaling factor, $\tau$, in units of time, in order to write the system in dimensionless form.

***Solution**:* Using conservation principles on each of the compartments, we obtain the system of ordinary differential equations

$$\begin{cases} \dfrac{dS}{dt} = -\beta SI; \\ \dfrac{dI}{dt} = \beta SI - \gamma I; \\ \dfrac{dR}{dt} = \gamma I. \end{cases} \tag{2}$$

It follows from the equations in (2) that $\beta$ has units of 1/[time × individual], while $\gamma$ has units of 1/time.

Next, use the change of variables in (1) and the Chain Rule to obtain from the first equation in (2) that

$$\begin{aligned} \frac{d\widehat{s}}{d\widehat{t}} &= \frac{d\widehat{s}}{dt} \cdot \frac{dt}{d\widehat{t}} \\ &= \frac{\tau}{N}\frac{dS}{dt} \\ &= -\frac{\tau}{N}\beta SI, \end{aligned}$$

so that, using (1) again,

$$\frac{d\widehat{s}}{d\widehat{t}} = -\beta\tau N\, \widehat{s}\, \widehat{i}. \tag{3}$$

Similar calculations for the second equation in (2) yield

$$\frac{d\widehat{i}}{d\widehat{t}} = \beta\tau N\, \widehat{s}\, \widehat{i} - \gamma\tau\widehat{i}; \tag{4}$$

and, for the third equation in (4),

$$\frac{d\widehat{r}}{d\widehat{t}} = \gamma\tau\widehat{i}. \tag{5}$$

Define the dimensionless parameter

$$\beta\tau N = R_o, \tag{6}$$

and set

$$\gamma\tau = 1,$$

so tat

$$\tau = \frac{1}{\gamma}, \tag{7}$$

and

$$R_o = \frac{\beta N}{\gamma}, \tag{8}$$

by virtue of (6).

Next, substitute (6) and (7) into the equations in (3), (4) and (5) to obtain the dimensionless system

$$\begin{cases} \dfrac{d\widehat{s}}{d\widehat{t}} = -R_o\,\widehat{s}\,\widehat{i}; \\ \\ \dfrac{d\widehat{i}}{d\widehat{t}} = R_o\,\widehat{s}\,\widehat{i} - \widehat{i}; \\ \\ \dfrac{d\widehat{r}}{d\widehat{t}} = \widehat{i}. \end{cases} \tag{9}$$

If we stipulate from the outset that $t$ is measured in units of $1/\gamma$ and $s$, $i$ and $r$ are measures in fractions of the total population, $N$, then the system in (9) can be written in simpler form as

$$\begin{cases} \dfrac{ds}{dt} = -R_o si; \\ \\ \dfrac{di}{dt} = R_o si - i; \\ \\ \dfrac{dr}{dt} = i, \end{cases}$$

which depends on the single dimensionless parameter, $R_o$, given in (8). □

2. **Modeling Traffic Flow.** Consider the initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + g'(u)\dfrac{\partial u}{\partial x} = 0; \\ \\ u(x,0) = f(x), \end{cases} \tag{10}$$

where

$$g(u) = u(1-u), \tag{11}$$

and the initial condition $f$ is given by

$$f(x) = \begin{cases} 1, & \text{if } x < -1; \\ \frac{1}{2}(1-x), & \text{if } -1 \leqslant x < 1; \\ 0, & \text{if } x \geqslant 1. \end{cases} \tag{12}$$

(a) Sketch the characteristic curves of the partial differential equation.

***Solution**:* The equation for the characteristic curves is given by

$$\frac{dx}{dt} = g'(u). \tag{13}$$

On characteristic curves, a solution, $u$, to the partial differential equation in (10) satisfies the ordinary differential equation

$$\frac{du}{dt} = 0,$$

which shows that $u$ is constant along characteristic curves. We write

$$u(x,t) = \varphi(k), \tag{14}$$

where $\varphi(k)$ is the constant value of $u$ on the characteristic indexed by $k$.

Using the value for $u$ in (14), the equation for the characteristic curves in (13) can be re–written as

$$\frac{dx}{dt} = g'(\varphi(k)). \tag{15}$$

Solving the differential equation in (15) yields the equation for the characteristic curves

$$x = g'(\varphi(k))t + k, \tag{16}$$

where the parameter $k$ corresponds to the value on the $x$–axis on which the characteristic curves meet the $x$–axis.

Next, solve for $k$ in (16) and substitute into (14) to obtain the expression

$$u(x,t) = \varphi(x - g'(u(x,t))t), \tag{17}$$

which gives a solution of the partial differential equation in (10) implicitly.

Using the initial condition in (10), we obtain from (17) that

$$\varphi(x) = f(x), \quad \text{for all } x \in \mathbb{R},$$

so that (17) can now be re–written as

$$u(x,t) = f(x - g'(u(x,t))t). \tag{18}$$

Accordingly, the equation for the characteristic curves in (16) can now be re–written as

$$x = g'(f(k))t + k, \tag{19}$$

so that the characteristic curves will be straight lines in the $xt$–plane of slope $1/g'(f(k))$ going through $(k,0)$ for $k \in \mathbb{R}$, where $g'(u)$ is obtained from (11) as

$$g'(u) = 1 - 2u. \tag{20}$$

For instance, using (20), (12) and (19) we get that the equations for the characteristic curves for $k \leqslant -1$ are given by

$$x = -t + k, \quad \text{for } k \leqslant -1. \tag{21}$$

The curves described by (21) are straight lines with slope $-1$ going through $(k,0)$, for $k \leqslant -1$. Some of these are pictured in Figure 2. Similarly, for

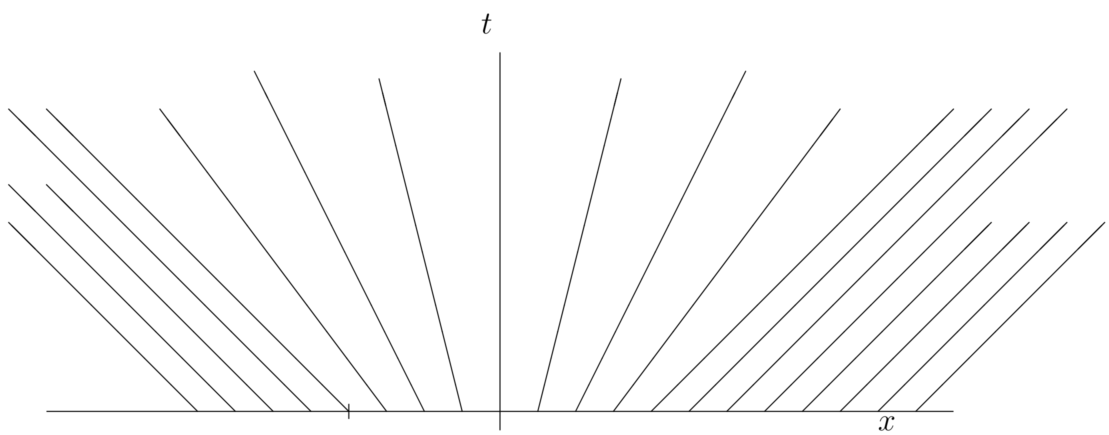


Figure 2: Characteristic Curves for Problem (10)

$k \geqslant 1$, the curves in (19) have equations

$$x = t + k, \quad \text{for } k \geqslant 1,$$

which are straight lines of slope 1 going through $(k, 0)$, for $k \geqslant 1$; some of these lines are also sketched in Figure 2.

For values of $k$ between $-1$ and 1, the slopes of the lines in (19) are given by $1/g'(f(k))$, where $f(k)$ ranges from 1 at $k = -1$, to 0 at $k = 1$; so, according to (20), the slopes of the lines are negative and increase in absolute value to $\infty$ as $k$ approaches 0. At $k = 0$, $f(k) = 1/2$, so that $g'(f(k)) = 0$, by virtue of (20), so that the characteristic curve will be $x = 0$, according to (19), or the $t$–axis. As $k$ ranges from 0 to 1, the characteristic curves fan out from the $t$–axis to the line $x = t + 1$. A few of these curves are shown in Figure 2. □

(b) Explain how the initial value problem can be solved in this case, and give a formula for $u(x, t)$.

***Solution**:* Since the characteristic curves do not intersect for $t > 0$, the initial value problem in (10) can always be solved by traveling back along the characteristic curves until they hit the $x$–axis at a point $(k, 0)$, and then reading the value of the initial density, $u(k, 0) = f(k)$, at that point. For example, if the point $(x, t)$ lies in the region $x < -t - 1$, we see from Figure 2 that the characteristic curve containing the point $(x, t)$ will meet the $x$–axis at some point $(k, 0)$ with $k < -1$; since, $f(k) = 1$ for $k < -1$, it follows from (18) that

$$u(x, t) = 1, \quad \text{for } x < -t - 1, \text{ and } t \geqslant 0. \tag{22}$$

Similarly, if $x \geqslant x + t$, then the characteristic curve containing $(x, t)$ will meet the $x$–axis at some point $(k, 0)$ with $k \geqslant 1$; since $f(k) = 0$ for $k \geqslant 1$, it follows from (18) that

$$u(x, t) = 0, \quad \text{for } x \geqslant t + 1, \text{ and } t \geqslant 0. \tag{23}$$

For $(x, t)$ lying in the region between the lines $x = -t-1$ and $x = t+1$, the characteristic curve containing the point will meet the $x$–axis at a point $(k, 0)$ with $-1 \leqslant k \leqslant 1$. Since $f(k) = \dfrac{1}{2}(1 - k)$ for those values of $k$, by (12), it follows from (18) that

$$u(x, t) = \frac{1}{2}[1 - (x - g'(u(x, t))t)], \quad \text{for } -t - 1 \leqslant x \leqslant t + 1. \tag{24}$$

Using (20), we can re–write (24) as

$$u(x, t) = \frac{1 - x + t}{2} - u(x, t)t, \quad \text{for } -t - 1 \leqslant x \leqslant t + 1. \tag{25}$$

Solving for $u(x,t)$ in (25) yields

$$u(x,t) = \frac{1-x+t}{2(1+t)}, \qquad \text{for } -t-1 \leqslant x \leqslant t+1. \tag{26}$$

Finally, putting together the results in (22), (23) and (26), we obtain the following formula for $u(x,t)$:

$$u(x,t) = \begin{cases} 1, & \text{for } x < -t-1; \\ \dfrac{1-x+t}{2(1+t)}, & \text{for } -t-1 \leqslant x \leqslant t+1; \\ 0, & \text{for } x > t+1, \end{cases}$$

for $t \geqslant 0$. □

3. **Traffic Flow at a Red light**. Let the initial condition in Problem 3 be given by

$$f(x) = \begin{cases} 1, & \text{if } x \leqslant 0; \\ 0, & \text{if } x > 0. \end{cases} \tag{27}$$

(a) Explain why this initial value problem models the situation at a traffic light before the light turns green.

***Solution**:* A car density of $u = 1$ corresponds to a maximum density, $\rho_{\text{max}}$, of bumper–to–bumper traffic, and zero speed. Thus, for $x \leqslant 0$ and $t = 0$, all drivers have stopped their vehicles and are waiting for the line to turn green in order to start moving. On the other side of the intersection, for $x > 0$, there are no vehicles; the car density is 0. □

(b) Sketch the characteristic curves of the partial differential equation.

***Solution**:* The equation for the characteristic curves of the PDE in (10) is

$$\frac{dx}{dt} = g'(u), \tag{28}$$

where

$$g'(u) = 1 - 2u. \tag{29}$$

On characteristic curves, a solution $u$ of the PDE in (10) satisfies the ODE

$$\frac{du}{dt} = 0, \tag{30}$$

which we can solve to get

$$u = \varphi(k), \tag{31}$$

where $\varphi$ is a differentiable function of a single variable, and $k$ is a real value indexing the characteristic curve on which the ODE in (30) holds true.

Next, substitute the expression for in (31) into (28) to get the ODE

$$\frac{dx}{dt} = g'(\varphi(k)),$$

which can be solved to yield

$$x = g'(\varphi(k))t + k \tag{32}$$

as the equation for the characteristic curves.

Now, solving for the left–most $k$ in (32) and substituting into (31) yields an implicit formula for $u(x, t)$,

$$u(x,t) = \varphi(x - g'(u(x,t))t), \tag{33}$$

where we have also used (31).

Using the initial condition in (10), we obtain from (33) that

$$\varphi(x) = f(x), \quad \text{for all } x \in \mathbb{R}.$$

We can therefore write the equation for the characteristic curves in (32) as

$$x = g'(f(k))t + k, \quad \text{for } k \in \mathbb{R}. \tag{34}$$

We next proceed to sketch the characteristic curves in (34).

For $k \leqslant 0$, we have that $f(k) = 1$; so that, according to (29), $g'(f(k)) = -1$ and, therefore, the characteristic curves for $k \leqslant 0$ are

$$x = -t + k, \quad \text{for } k \leqslant 0,$$

where we have used (34). These are sketched in Figure 3.

Similarly, for $k > 0$ we have that $f(k) = 0$ and, therefore, $g'(f(k)) = 1$; so that the characteristic curves for $k > 0$ are

$$x = t + k.$$

These are also sketched in Figure 3. □

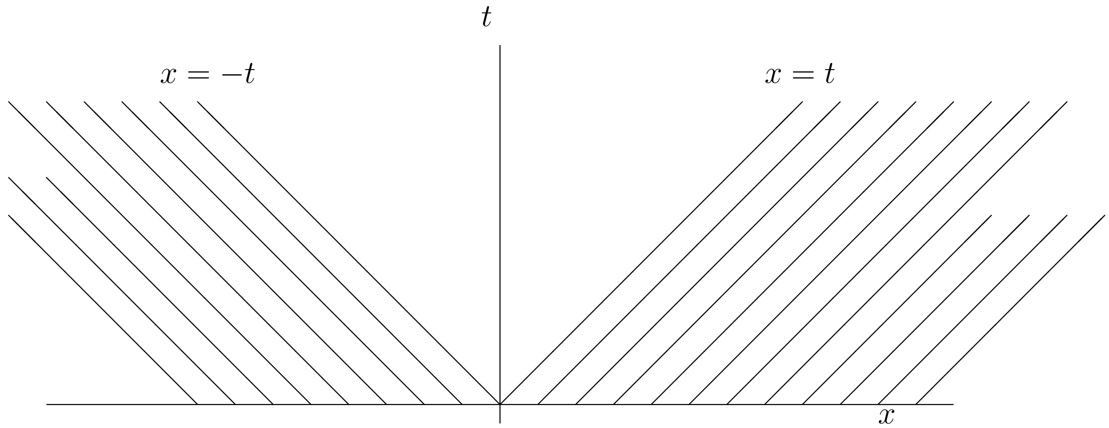


Figure 3: Characteristic Curves in (34)

(c) Explain why a shock wave solution doe not develop at $t = 0$.

***Solution**:* The sketch in Figure 3 shows that the characteristic curves of the PDE in (10) with the initial condition in (27) do not intersect for $t > 0$. Hence, no shock wave solution develops at $t = 0$. □

(d) Look for a solution to the equation of the form

$$u(x,t) = \varphi\left(\frac{x}{t}\right), \quad \text{for } -t < x < t, \quad \text{and} \quad t > 0,$$

where $\varphi$ is a differentiable function of a single variable.

*Suggestion:* Introduce a new variable $\eta = \dfrac{x}{t}$, and compute $\dfrac{d\varphi}{d\eta}$.

***Solution**:* The sketch of the characteristic curves in Figure 3 shows that a solution of the IVP in (10) with the initial condition in (27) can be obtained for $x < -t$ or $x > t$, and $t > 0$, by travelling back along the characteristic curves to the $x$–axis. In fact, $u(x,t) = 1$ for $x \leqslant -t$ and $u(x,t) = 0$ for $x > t$. It remains therefore to find a formula for $u(x,t)$ for $-t < x \leqslant t$ and $t > 0$. To do so, we look for a solution of the form

$$u(x,t) = \varphi\left(\frac{x}{t}\right), \quad \text{for } t > 0, \tag{35}$$

where $\varphi$ is a differentiable function of a single variable.

In order for the function $u$, given in (35), to be a solution of the equation in (10) it must satisfy the partial differential equation

$$\frac{\partial u}{\partial t} + (1 - 2u)\frac{\partial u}{\partial x} = 0, \quad \text{for } t > 0. \tag{36}$$

Thus, we need to compute the partial derivatives of $u$ in terms of $\varphi$ and substitute them in the equation in (36). In order to do this, we introduce the variable

$$\eta = \frac{x}{t}; \tag{37}$$

so that, according to (35),

$$u(x,t) = \varphi(\eta); \tag{38}$$

so that, applying the Chain Rule,

$$\frac{\partial u}{\partial t} = \varphi'(\eta) \cdot \frac{\partial \eta}{\partial t} = \varphi'(\eta)\left(-\frac{x}{t^2}\right),$$

or

$$\frac{\partial u}{\partial t} = -\frac{1}{t}\eta\varphi'(\eta), \qquad \text{for } t > 0. \tag{39}$$

Similarly, differentiating with respect to $x$,

$$\frac{\partial u}{\partial t} = \varphi'(\eta) \cdot \frac{\partial \eta}{\partial x} = \varphi'(\eta)\left(\frac{1}{t}\right),$$

or

$$\frac{\partial u}{\partial x} = \frac{1}{t}\varphi'(\eta), \qquad \text{for } t > 0. \tag{40}$$

Now, substituting (40), (39) and (38) into (36) yields

$$-\frac{1}{t}\eta\varphi'(\eta) + (1 - 2\varphi(\eta))\,\frac{1}{t}\varphi'(\eta) = 0,$$

which simplifies to

$$(1 - \eta - 2\varphi(\eta))\varphi'(\eta) = 0, \qquad \text{for } t > 0. \tag{41}$$

Now, it follows from (41) that, either

$$\varphi'(\eta) = 0, \tag{42}$$

or

$$\varphi(\eta) = \frac{1-\eta}{2}. \tag{43}$$

If (42) holds true for all $\eta$, then $\varphi$ would have to be constant. This would say, in view of (38), that $u(x,t)$ would have to be constant. However, this conclusion would not agree with the fact that $u(x,t)$ is 1 on the region in the $xt$–plane corresponding to characteristic curves emanating from $(k, 0)$ with $k < 0$, and $u(x, t)$ is 0 in the region corresponding to characteristic curves emanating from $(k, 0)$ for $k > 0$ (see Figure 3). Hence, it must be the case that $\varphi$ is as given in (43). We therefore have, according to that (38) and (37) that

$$u(x,t) = \frac{1 - \dfrac{x}{t}}{2}, \qquad \text{for } -t < x \leqslant t, \qquad \text{and} \qquad t > 0,$$

or

$$u(x,t) = \frac{t - x}{2t}, \qquad \text{for } -t < x \leqslant t, \qquad \text{and} \qquad t > 0.$$

Combing this result with the information obtained from the characteristic sketched in Figure 3 we get that

$$u(x,t) = \begin{cases} 1, & \text{if } x \leqslant -t; \\ \\ \dfrac{t-x}{2t}, & \text{if } -t < x \leqslant t; \\ \\ 0, & \text{if } x > t, \end{cases}$$

for $t > 0$. □