Solutions to Review Problems for Exam #2

1. **The Poisson Random Process Revisited**. We saw in class and in the lecture notes online how to define a Poisson random process, $\{M(t) \mid t \geqslant 0\}$, to model occurrences of events at random points in time (e.g., occurrences of mutations in a bacterial colony). Here $M(t)$ counts the number of occurrences in the time interval $[0, t]$. This continuous–time random process may also be defined as one satisfying the following axioms:

   (i) $M(0) = 0$.

   (ii) The number of events that occur in disjoint time intervals are independent; in symbols, for $t_1 < t_2 \leqslant t_3 < t_4$,

   $$M(t_2) - M(t_1) \text{ and } M(t_4) - M(t_3) \text{ are independent random variables.}$$

   (iii) The number of occurrences within a time interval depends only on the length of the time interval; in symbols, for all $t, s > 0$, $M(t+s) - M(t)$ depends only on s, so that

   $$\Pr[M(t+s) - M(t) = k] = \Pr[M(s) - M(0) = k], \quad \text{for all } k.$$

   (iv) $\Pr[M(\Delta t) = 1] = \lambda \Delta t + o(\Delta t)$.

   (v) $\Pr[M(\Delta t) \geqslant 2] = o(\Delta t)$.

   The notation $o(h)$ in (iv) and (v) is defined as follows: We say that an expression, $f(h)$, is $o(h)$ as $h \to 0$ iff $\lim_{h\to 0} \dfrac{f(h)}{h} = 0$.

   The constant λ in (iv) is called the rate of the process.

   Set

   $$P_m(t) = \Pr[M(t) = m], \quad \text{for } m = 0, 1, 2, 3 \ldots, \text{ and } t \geqslant 0. \tag{1}$$

   Use the axioms (i)–(v) to prove the following assertions.

   (a) For $t, s > 0$,

   $$P_0(t+s) = P_0(t) \cdot P_0(s). \tag{2}$$

   Suggestion: Consider the event $[M(t) = 0, M(t+s) - M(t) = 0]$ or

   $$[M(t) = 0] \cap [M(t+s) - M(t) = 0].$$

***Solution**:* It follows from axiom (ii) that the random variables

$$M(t) - M(0) \qquad \text{and} \qquad M(t+s) - M(t)$$

are independent; consequently, the events

$$(M(t) = 0) \qquad \text{and} \qquad (M(t+s) - M(t) = 0)$$

are independent (here we have also used axiom (i)); thus,

$$\Pr(M(t) = 0, M(t+s) - M(t) = 0) = \Pr(M(t) = 0) \cdot \Pr(M(t+s) - M(t) = 0),$$

or

$$\Pr(M(t+s) = 0) = \Pr(M(t) = 0) \cdot \Pr(M(t+s) - M(t) = 0).$$

Hence, using the definition of P_o in (1),

$$P_o(t+s) = P_o(t) \cdot \Pr(M(t+s) - M(t) = 0).$$

Thus, using axiom (iii),

$$P_o(t+s) = P_o(t) \cdot \Pr(M(s) - M(0) = 0),$$

or, in view of axiom (i),

$$P_o(t+s) = P_o(t) \cdot \Pr(M(s) = 0),$$

from which (2) follows by virtue of the definition of P_o in (1). □

(b) Use (2) and axioms (iv) and (v) to derive the differential equation

$$\frac{dP_0}{dt} = -\lambda P_0(t). \tag{3}$$

Suggestion: Verify that

$$P_o(t + \Delta t) - P_0(t) = -\lambda P_o(t)\Delta t + o(\Delta t). \tag{4}$$

***Solution**:* Substituting Δt for s in (2), we get

$$P_o(t + \Delta t) = P_o(t) \cdot P_o(\Delta t), \tag{5}$$

where

$$\begin{aligned} P_o(\Delta t) &= \Pr(M(\Delta t) = 0) \\ &= 1 - \Pr(M(\Delta t) \geqslant 1) \\ &= 1 - \Pr(M(\Delta t) = 1) - \Pr(M(\Delta t) \geqslant 2); \end{aligned}$$

so that, in view of (iv) and (v),

$$P_o(\Delta t) = 1 - \lambda \Delta t + o(\Delta t). \tag{6}$$

Consequently, substituting (6) into (5),

$$P_o(t + \Delta t) = P_o(t)(1 - \lambda \Delta t + o(\Delta t)),$$

from which we get

$$P_o(t + \Delta t) = P_o(t) - \lambda P_o(t)\Delta t + o(\Delta t),$$

which yields (4).

Next, divide both sides of (4) by $\Delta t \neq 0$ to get

$$\frac{P_o(t + \Delta t) - P_o(t)}{\Delta t} = -\lambda P_o(t) + \frac{o(\Delta t)}{\Delta t}. \tag{7}$$

It follows from (7) and the definition of "o" that P_o is differentiable at every $t > 0$, and its derivative is given by

$$\frac{dP_o(t)}{dt} = -\lambda P_o(t),$$

which is (3). □

(c) Solve the differential equation in (3) subject to the initial condition in (i) to obtain and expression for $P_0(t)$ for all $t \geqslant 0$.

***Solution**:* The general solution of the differential equation in (3) is

$$P_o(t) = Ce^{-\lambda t}, \quad \text{for } t \geqslant 0, \tag{8}$$

for arbitrary constant C.

Observe that

$$P_o(0) = \Pr(M(0) = 0) = 1,$$

by virtue of axiom (i). It then follows from (8) that $C = 1$; consequently,

$$P_o(t) = e^{-\lambda t}, \quad \text{for } t \geqslant 0, \tag{9}$$

□

(d) Let T_1 denote the time of the first occurrence, and, for $n \geqslant 2$, let T_n denote the time elapsed between the $(n-1)^{\text{st}}$ occurrence and the the n^{th} occurrence. The sequence (T_n) is called the sequence of interarrival times. Give the distribution for each of the random variables T_n.

Suggestion: We have already done the derivation of the distribution for T_1 in the class notes and assignments. Please, present the derivation here as well.

For $n = 2$, consider the conditional probabilities

$$\Pr[T_2 > s + t \mid T_1 = s] \text{ and } \Pr[M(s+t) - M(s) = 0 \mid M(s) = 1].$$

Solution*:* We first compute the cumulative distribution function of T_1,

$$\begin{aligned} F_{T_1}(t) &= \Pr(T_1 \leqslant t), \quad \text{for } t \geqslant 0; \\ &= 1 - \Pr(T_1 > t) \\ &= 1 - \Pr(M(t) = 0), \end{aligned}$$

since $T_1 > t$ implies that no event has occurred at time t. Consequently, using (9)

$$F_{T_1}(t) = 1 - e^{-\lambda t}, \qquad \text{for } t \geqslant 0.$$

Thus, the probability density function of T_1 is given by

$$f_{T_1}(t) = \begin{cases} \lambda e^{-\lambda t}, & \text{if } t > 0; \\ 0, & \text{if } t \leqslant 0. \end{cases} \tag{10}$$

Hence, T_1 has an exponential distribution with parameter $1/\lambda$.

Next, we find the distribution of T_2. To do this, we first compute conditional probability

$$\Pr(T_2 \leqslant s + t \mid T_1 = s) = 1 - \Pr(T_2 > s + t \mid T_1 = s), \tag{11}$$

since the second occurrence will happen at some time t after the first occurrence at $T_1 = s$.

Observe that the event $(T_2 > s + t \mid T_1 = s)$ is the same as the event that there are no new occurrances after the first occurrence at $T_1 = s$ in the time interval $(s, s+t)$; that is, the event $(M(s+t) - M(s) = 0)$ conditioned on the event $M(s) = 1$. We therefore have that

$$\Pr(T_2 > s + t \mid T_1 = s) = \Pr(M(s+t) - M(s) = 0 \mid M(s) = 1)$$

or

$$\Pr(T_2 > s+t \mid T_1 = s) = \Pr(M(s+t) - M(s) = 0 \mid M(s) - M(0) = 1),$$

by virtue of axiom (i). Thus, by axiom (ii)

$$\Pr(T_2 > s+t \mid T_1 = s) = \Pr(M(s+t) - M(s) = 0),$$

and, by axiom (iii),

$$\Pr(T_2 > s+t \mid T_1 = s) = \Pr(M(t) - M(0) = 0),$$

or

$$\Pr(T_2 > s+t \mid T_1 = s) = \Pr(M(t) = 0), \tag{12}$$

where we have used axiom (i) again.
Using the definition of P_o in (1) we obtain from (13) that

$$\Pr(T_2 > s+t \mid T_1 = s) = P_o(t);$$

so that, in view of (9),

$$\Pr(T_2 > s+t \mid T_1 = s) = e^{-\lambda t}, \qquad \text{for } t \geqslant 0. \tag{13}$$

Combining (11) and (13)

$$\Pr(T_2 \leqslant s+t \mid T_1 = s) = 1 - e^{-\lambda t}, \quad \text{for } t \geqslant 0,$$

which shows that $T_2 \sim \text{Exponential}(1/\lambda)$; that is, T_2 and T_1 have the same distribution.
A similar argument to that used for T_2 shows that $T_k \sim \text{Exponential}(1/\lambda)$ for $k > 2$. Consequently,

$$T_k \sim \text{Exponential}(1/\lambda), \quad \text{for } k = 1, 2, 3, \ldots$$

□

(e) Let S_n denote the time of occurrence of the n^{th} event, so that

$$S_n = \sum_{k=1}^{n} T_k, \quad \text{for } n = 1, 2, 3, \ldots \tag{14}$$

Show that, for each $n = 1, 2, 3, \ldots$, S_n is a continuous random variable with density function given by

$$f_{S_n}(s) = \lambda e^{-\lambda s}\frac{(\lambda s)^{n-1}}{(n-1)!}, \quad \text{for } n = 1, 2, 3, \ldots \tag{15}$$

Suggestion: Proceed by induction on n. The base case, $n = 1$, has already been established. For the case $n = 2$, so that $S_2 = T_1 + T_2$, use the fact that, since T_1 and T_2 are independent random variables, the distribution of S_2 is given by the convolution formula

$$f_{S_n}(s) = f_{T_1} * f_{T_2}(s) = \int_{-\infty}^{\infty} f_{T_1}(\tau) f_{T_2}(s - \tau)\, d\tau \tag{16}$$

Note that the convolution formula in (16) applies to any sum of independent, continuous random variables.

***Solution**:* We first note that the random variables, T_k, obtained in the previous part are mutually independent. This is a consequence of axiom (ii), since the occurrence of an event after any number of events have occurred is in independent of how many events have occurred previously.

First, note that $S_1 = T_1$, according to (14), and T_1 has the distribution function given in (10), which the distribution function given in (15) for the case $n = 1$.

Next, we consider the case $n = 2$. In this case, using (14), $S_2 = T_1 + T_2$, were T_1 and T_2 are independent, Exponential$(1/\lambda)$ random variables. We can therefore use the formula in (16) to find the distribution of S_2:

$$f_{S_2}(s) = \int_{-\infty}^{\infty} f_{T_1}(\tau) f_{T_2}(s - \tau)\, d\tau,$$

where the distribution functions of T_1 and T_2 are both given by (10); consequently,

$$\begin{aligned} f_{S_2}(s) &= \int_0^s \lambda e^{-\lambda\tau} \lambda e^{-\lambda(s-\tau)}\, d\tau \\ &= \lambda^2 e^{-\lambda s} \int_0^s d\tau \\ &= \lambda^2 e^{-\lambda s} s, \end{aligned}$$

which we can rewrite as

$$f_{S_2}(s) = \lambda e^{-\lambda s}(\lambda s), \qquad \text{for } s > 0,$$

which is (16) for the case $n = 2$.

We now proceed by induction on n, assuming the statement is true for n, and showing that it must be true for $n + 1$. We therefore consider

$$S_{n+1} = S_n + T_{n+1},$$

where S_n and T_{n+1} are independent random variables with distributions given by (14), for $s > 0$, and (10), respectively. Thus, using the convolution formula in (16), this time for S_n and T_{n+1} in place of T_1 and T_2,

$$\begin{aligned} f_{S_{n+1}}(s) &= \int_{-\infty}^{\infty} f_{S_n}(\tau) f_{T_{n+1}}(s-\tau)\, d\tau \\ &= \int_0^s \lambda e^{-\lambda\tau} \frac{(\lambda\tau)^{n-1}}{(n-1)!} \lambda e^{-\lambda(s-\tau)}\, d\tau \\ &= \frac{\lambda^{n+1}}{(n-1)!} e^{-\lambda s} \int_0^s \tau^{n-1}\, d\tau \\ &= \frac{\lambda^{n+1}}{(n-1)!} e^{-\lambda s} \frac{s^n}{n}, \end{aligned}$$

which we can rewrite as

$$f_{S_{n+1}}(s) = \lambda e^{-\lambda s} \frac{(\lambda s)^n}{n!}, \qquad \text{for } s > 0,$$

which is (15) for $n+1$ in place of n. □

(f) Use the result in (15) to derive the formula

$$P_m(t) = \frac{(\lambda t)^m}{m!}\, e^{-\lambda t}, \qquad \text{for } m = 0, 1, 2, 3, \ldots, \text{ and } t \geqslant 0. \tag{17}$$

Suggestion: Consider the events

$$[M(t) \geqslant n] \quad \text{and} \quad [S_n \leqslant t]$$

and note that

$$[M(t) = n] = [n \leqslant M(t) < n+1]$$

***Solution**:* We compute

$$\Pr(M(t) = m) = \Pr(m \leqslant M(t) < m+1). \tag{18}$$

Note that the $S_n \leqslant t$ if and only if there have been at least n occurrences at in the interval $[0, t]$; consequently

$$(M(t) \geqslant n) \text{ and } (S_n \leqslant t) \text{ are the same events.} \tag{19}$$

Similarly,

$$(M(t) \geqslant n+1) \text{ and } (S_{n+1} \leqslant t) \text{ are the same events.} \tag{20}$$

Now, it follows from

$$(M(t) \geqslant m) = (m \leqslant M(t) < m+1) \cup (M(t) \geqslant m+1)$$

that

$$\Pr(M(t) \geqslant m) = \Pr(m \leqslant M(t) < m+1) + \Pr(M(t) \geqslant m+1),$$

from which we get that

$$\Pr(m \leqslant M(t) < m+1) = \Pr(M(t) \geqslant m) - \Pr(M(t) \geqslant m+1);$$

consequently, in view of (19) and (20)

$$\Pr(m \leqslant M(t) < m+1) = \Pr(S_m \leqslant t) - \Pr(S_{m+1} \leqslant t). \tag{21}$$

Next, use the distribution function in (15) to compute

$$\Pr(S_m \leqslant t) = \int_0^t \lambda e^{-\lambda s} \frac{(\lambda s)^{m-1}}{(m-1)!}\, ds,$$

which we can evaluate using integration by parts.
Set

$$u = \lambda e^{-\lambda s} \quad \text{and} \quad dv = \frac{(\lambda s)^{m-1}}{(m-1)!};$$

so that,

$$du = -\lambda^2 e^{-\lambda s}\, ds \quad \text{and} \quad v = \frac{1}{\lambda}\frac{(\lambda s)^m}{m!}.$$

Thus,

$$\Pr(S_m \leqslant t) = \frac{(\lambda s)^m}{m!} e^{-\lambda s}\Big|_0^t + \int_0^t \lambda e^{-\lambda s} \frac{(\lambda s)^m}{m!}\, ds,$$

or

$$\Pr(S_m \leqslant t) = \frac{(\lambda t)^m}{m!} e^{-\lambda t} + \Pr(S_{m+1} \leqslant t), \tag{22}$$

Comparing (21) and (22) we see that

$$\Pr(m \leqslant M(t) < m+1) = \frac{(\lambda t)^m}{m!} e^{-\lambda t}, \tag{23}$$

for $m = 0, 1, 2, 3, \ldots$, and $t \geqslant 0$.
The statement in (17) now follows from (18) and (23). □

(g) Suppose that exactly one event has occurred in the time interval $[0, \tau]$. We consider the time of occurrence, T_1, of that event. Compute the conditional probability

$$\Pr[T_1 \leqslant t \mid M(\tau) = 1], \text{ for } 0 < t < \tau.$$

Suggestion: Consider the events

$$[T_1 \leqslant t, M(\tau) = 1] \quad \text{and} \quad [M(t) = 1, M(\tau) - M(t) = 0],$$

for $0 < t < \tau$.

Solution: By the definition of conditional probability,

$$\Pr(T_1 \leqslant t \mid M(\tau) = 1) = \frac{\Pr(T_1 \leqslant t \mid M(\tau) = 1)}{\Pr(M(\tau) = 1)}, \tag{24}$$

where, according to (17),

$$\Pr(M(\tau) = 1) = \lambda\tau e^{-\lambda\tau}. \tag{25}$$

Note that, for $0 < t < \tau$,

$$(T_1 \leqslant t, M(\tau) = 1) \text{ and } (M(t) = 1, M(\tau) - M(t) = 0),$$

since the first occurrence in $(0, t]$ and no occurrence in $(t\tau]$ is the same as exactly one occurrence in $(0, \tau]$ and the time of the first occurrence and t coming after the first occurrence. Consequently,

$$\Pr(T_1 \leqslant t, M(\tau) = 1) = \Pr(M(t) = 1, M(\tau) - M(t) = 0);$$

so that, using axioms (i), (ii) and (iii),

$$\Pr(T_1 \leqslant t, M(\tau) = 1) = \Pr(M(t) = 1) \cdot \Pr(M(\tau - t) = 0),$$

or

$$\Pr(T_1 \leqslant t, M(\tau) = 1) = \lambda t e^{-\lambda t} \cdot e^{-\lambda(\tau - t)},$$

or

$$\Pr(T_1 \leqslant t, M(\tau) = 1) = \lambda t \cdot e^{-\lambda\tau}. \tag{26}$$

Combining (24), (25) and (26) we see that

$$\Pr(T_1 \leqslant t \mid M(\tau) = 1) = \frac{t}{\tau}, \quad \text{for } 0 < t < \tau. \tag{27}$$

□

2. **The Error function**, Erf: $\mathbb{R} \to \mathbb{R}$, is defined by

$$\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2}\, ds, \qquad \text{for } x \in \mathbb{R}. \tag{28}$$

Use the fact that

$$\int_0^\infty e^{-s^2}\, ds = \frac{\sqrt{\pi}}{2} \tag{29}$$

to deduce that

(a) $\lim\limits_{x\to\infty} \text{Erf}(x) = 1$; and

(b) $\lim\limits_{x\to-\infty} \text{Erf}(x) = -1$.

***Solution**:*

(a) The expression in (29) is equivalent to

$$\lim_{x\to\infty} \int_0^x e^{-s^2}\, ds = \frac{\sqrt{\pi}}{2},$$

or

$$\lim_{x\to\infty} \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2}\, ds = 1,$$

from which we get that

$$\lim_{x\to\infty} \text{Erf}(x) = 1, \tag{30}$$

which was to be shown.

(b) Since the map $s \mapsto e^{-s^2}$ is even, it follows that, for any $x < 0$,

$$\int_x^0 e^{-s^2}\, ds = \int_0^{-x} e^{-s^2}\, ds,$$

so that

$$-\int_0^x e^{-s^2}\, ds = \int_0^{-x} e^{-s^2}\, ds,$$

which yields

$$\text{Erf}(x) = -\text{Erf}(-x), \qquad \text{for all } x < 0.$$

Thus,

$$\lim_{x\to-\infty} \text{Erf}(x) = -\lim_{x\to-\infty} \text{Erf}(-x) \tag{31}$$

Making the change of variables $y = -x$ on the right–hand side of (31) we see that

$$\lim_{x\to-\infty} \text{Erf}(x) = -\lim_{y\to\infty} \text{Erf}(y). \tag{32}$$

Combining (30) and (32)

$$\lim_{x\to-\infty} \text{Erf}(x) = -1, \tag{33}$$

which was to be shown.

□

3. **Solving the Heat Equation**. In this problem we compute a solution of the initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2}, & x \in \mathbb{R},\ t > 0; \\ u(x,0) = f(x), & x \in \mathbb{R}, \end{cases} \tag{34}$$

where

$$f(x) = \begin{cases} 1, & \text{if } x \leqslant 0; \\ 0, & \text{if } x > 0. \end{cases} \tag{35}$$

(a) Use the heat kernel to give a solution of the IVP (34).

***Solution**:* A candidate for a solution is given by

$$u(x,t) = \int_{-\infty}^{\infty} p(x-y,t) f(y)\, dy, \quad \text{for } x \in \mathbb{R} \text{ and } t > 0, \tag{36}$$

where

$$p(x,t) = \frac{e^{-x^2/4Dt}}{\sqrt{4\pi Dt}}, \quad \text{for } x \in \mathbb{R} \text{ and } t > 0. \tag{37}$$

Use (35) and (37) to obtain from (36) that

$$u(x,t) = \int_{-\infty}^{0} \frac{e^{-(x-y)^2/4Dt}}{\sqrt{4\pi Dt}}\, dy, \quad \text{for } x \in \mathbb{R} \text{ and } t > 0. \tag{38}$$

Make the change variables $s = \dfrac{x-y}{\sqrt{4Dt}}$ in (38) to obtain

$$u(x,t) = -\frac{1}{\sqrt{\pi}} \int_{\infty}^{x/\sqrt{4Dt}} e^{-s^2}\, ds, \quad \text{for } x \in \mathbb{R} \text{ and } t > 0,$$

or

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{x/\sqrt{4Dt}}^{\infty} e^{-s^2}\, ds, \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0,$$

or

$$u(x,t) = \frac{1}{2}\left(\frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-s^2}\, ds - \frac{2}{\sqrt{\pi}} \int_{0}^{x/\sqrt{4Dt}} e^{-s^2}\, ds \right),$$

for$x \in \mathbb{R}$ and $t > 0$; so that, using (29) and the definition of the Error function in (28),

$$u(x,t) = \frac{1}{2}\left[1 - \text{Erf}\left(\frac{x}{\sqrt{4Dt}}\right)\right], \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0. \tag{39}$$

□

(b) Use a mathematical software package to sketch the graph of $x \mapsto u(x,t)$ for several values of $t > 0$, where $u(x,t)$ is the solution of the initial value problem (34) with initial condition in (35) obtained in part (a).

***Solution**:* Set $4D = 1$ in (39) to get

$$u(x,t) = \frac{1}{2}\left[1 - \text{Erf}\left(\frac{x}{\sqrt{t}}\right)\right], \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0. \tag{40}$$

Figure shows sketches of the graphs of $y = u(x,t)$ for $t = 0.1, 1, 10$ □

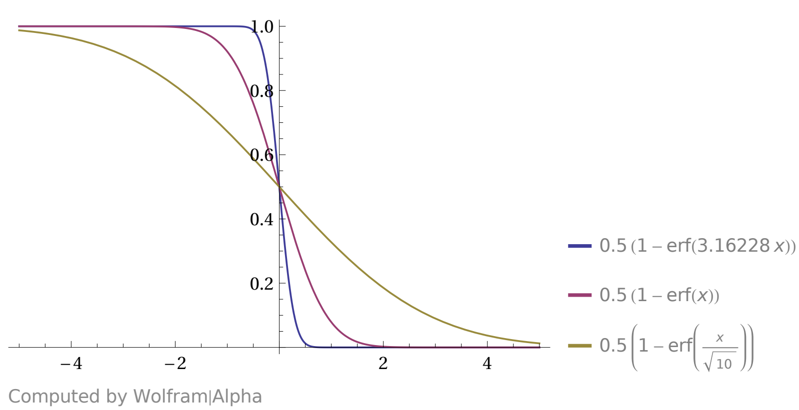

Figure 1: Sketch of Graph of $y = u(x,t)$ for $t = 0.1, 1, 10$

(c) Let $u(x,t)$ be the solution to the initial value problem (34) with initial condition in (35) obtained in part (a). Compute the following

(i) $\lim\limits_{t\to 0^+} u(x,t)$, for $x=0$ and for $x\neq 0$.
Solution: Let $u(x,t)$ be as given in (39); then, for $x=0$ we get

$$\lim_{t\to 0^+} u(x,t) = \frac{1}{2}.$$

For $x\neq 0$, we consider two possibilities: $x<0$ and $x>0$.
If $x<0$, we obtain

$$\lim_{t\to 0^+} u(x,t) = \lim_{t\to 0^+} \frac{1}{2}\left[1-\text{Erf}\left(\frac{x}{\sqrt{4Dt}}\right)\right] = \frac{1}{2}\left[1-(-1)\right] = 1.$$

If $x>0$,

$$\lim_{t\to 0^+} u(x,t) = \lim_{t\to 0^+} \frac{1}{2}\left[1-\text{Erf}\left(\frac{x}{\sqrt{4Dt}}\right)\right] = \frac{1}{2}\left[1-1\right] = 0.$$

□

(ii) $\lim\limits_{x\to 0} u(x,t)$, for all $t>0$.
Solution: Compute

$$\lim_{x\to 0} u(x,t) = \lim_{x\to 0} \frac{1}{2}\left[1-\text{Erf}\left(\frac{x}{\sqrt{4Dt}}\right)\right] = \frac{1}{2}\left[1-0\right] = \frac{1}{2}.$$

□

(d) Let $u(x,t)$ be the solution of the initial value problem (34) with initial condition in (35) obtained in part (a). Compute the following

(i) $\lim\limits_{t\to\infty} u(x,t)$, for $x=0$ and for $x\neq 0$.
Solution: We consider two cases: (i) $x=0$, and (ii) $x\neq 0$.

(i) If $x=0$,

$$\lim_{t\to\infty} u(x,t) = \frac{1}{2}.$$

(ii) If $x\neq 0$,

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} \frac{1}{2}\left[1-\text{Erf}\left(\frac{x}{\sqrt{4Dt}}\right)\right] = \frac{1}{2}\left[1-0\right] = \frac{1}{2}.$$

Thus, in both cases $\lim\limits_{t\to\infty} u(x,t) = \dfrac{1}{2}$. □

(ii) $\lim_{x\to\infty} u(x,t)$, for all $t > 0$.
Solution: Compute

$$\lim_{x\to\infty} u(x,t) = \lim_{x\to\infty} \frac{1}{2}\left[1 - \text{Erf}\left(\frac{x}{\sqrt{4Dt}}\right)\right] = \frac{1}{2}\,[1-1] = 0.$$

□

(iii) $\lim_{x\to-\infty} u(x,t)$, for all $t > 0$.
Solution: Compute

$$\lim_{x\to-\infty} u(x,t) = \lim_{x\to-\infty} \frac{1}{2}\left[1 - \text{Erf}\left(\frac{x}{\sqrt{4Dt}}\right)\right] = \frac{1}{2}\,[1-(-1)] = 1.$$

□