Solutions to Assignment #10

1. Let $N = N(t)$ denote the number of radioactive isotopes of an element in a sample at time t. Assume that the number of atoms that decay in a unit of time is a fraction, λ, of the number of isotopes present at the time.

   (a) Explain how the differential equation for $N = N(t)$,

$$\frac{dN}{dt} = -\lambda N, \tag{1}$$

   is derived. In particular, explain the minus sign on the right–hand side of equation (1).

   ***Solution**:* We may derive (1) by applying the conservation equation

$$\frac{dN}{dt} = \text{Rate of isotopes in} - \text{Rate of isotopes out}. \tag{2}$$

   Assuming that the isotopes are decaying by emitting radiation and turning into other atoms, and that no more radioactive isotopes are coming into the sample, we have that

$$\text{Rate of isotopes in} = 0, \tag{3}$$

   and

$$\text{Rate of isotopes out} = \lambda N. \tag{4}$$

   Substituting (3) and (4) into (2) yields (1). □

   (b) Assuming that λ is constant, solve the differential in (1) subject to the initial condition $N(t_o) = N_o$.

   ***Solution**:* The solution to the initial value problem

$$\begin{cases} \dfrac{dN}{dt} = -\lambda N; \\ N(t_o) = N_o, \end{cases}$$

   is

$$N(t) = N_o e^{-\lambda(t-t_o)}, \qquad \text{for all } t \geqslant t_o. \tag{5}$$

   □

   (c) Use the solution to the equation to the differential equation in (1) that you found in part (b) to find the length of time, $\tau_{1/2}$, from t_o at which the number of radioactive isotopes left in the sample is half of N_o. The time $\tau_{1/2}$ is called the half–life of the isotope.

***Solution**:* We want to find $\tau > 0$ such that

$$N(t_o + \tau) = \frac{1}{2} N_o. \tag{6}$$

Thus, using (5), we obtain from (6) that

$$N_o e^{-\lambda\tau} = \frac{1}{2} N_o,$$

which can be solved for τ to yield

$$\tau = \frac{\ln 2}{\lambda},$$

so that

$$\tau_{1/2} = \frac{\ln 2}{\lambda}. \tag{7}$$

□

2. (This problem and the next are based on Problem 12 on page 199 in *Essential Calculus with Applications* by Richard A. Silverman). When neutrons resulting from cosmic rays interactions in the upper atmosphere collide with Nitrogen molecules, Carbon–14, denoted by ${}^{14}_{6}C$, is produced in a nuclear reaction. ${}^{14}_{6}C$ is a radioactive isotope of carbon, ${}^{12}_{6}C$, that has a half–life of about 5700 years; thus, $\tau_{1/2} = 5700$ years for ${}^{14}_{6}C$. Radioactive isotopes ${}^{14}_{6}C$ combine with oxygen to form radioactive carbon dioxide (CO_2), or radiocarbon. Radiocarbon is absorbed by plants during photosynthesis, and then by plant–eating animals. As long as plants and animals are alive, they take in fresh radiocarbon. When they die, the process of taking in fresh radiocarbon stops and the radiocarbon begins to decay.

   Suppose that a tree died at time t_o in the past. Assume that the content of radiocarbon in a sample of the tree's heartwood at the time of death is N_o. Give the amount of radiocarbon that remains in the sample τ years later in terms of τ and $\tau_{1/2}$.

   ***Solution**:* Solving for λ in (7) we obtain that

$$\lambda = \frac{\ln 2}{\tau_{1/2}}. \tag{8}$$

   Substituting (8) into (5) yields

$$N(t) = N_o \exp\left(-\frac{\ln 2}{\tau_{1/2}}(t - t_o)\right), \quad \text{for all } t \geqslant t_o. \tag{9}$$

Next, substitute $t = t_o + \tau$ into (9) to yield

$$N(t_o + \tau) = N_o \exp\left(-\frac{\ln 2}{\tau_{1/2}}\tau\right), \tag{10}$$

which is the amount of radiocarbon that remains in the sample τ years later. □

3. Let $n(t)$ denote the fraction of ${}^{14}_{6}C$ to ${}^{12}_{6}C$ in a sample at time t; that is, fraction of the radioactive isotope, carbon–14, to that of the stable isotope, carbon–12 present in a sample at time t. Since, the stable form of carbon does not decay, the amount of ${}^{12}_{6}C$ in a sample should remain constant throughout the years.

   (a) Use your result from Problem 2 to obtain an expression in $n(t_o+\tau)$, τ years after the tree died, in terms of n_o, the fraction of carbon–14 to carbon–12 in the sample at the time of death, t_o.

   ***Solution**:* Divide the equation in (10) by the number of carbon–12 isotopes in the sample to obtain

$$n(t_o + \tau) = n_o \exp\left(-\frac{\ln 2}{\tau_{1/2}}\tau\right). \tag{11}$$

□

   (b) Assuming that n_o is the same as the one for all living organisms at present time, give an estimate of the age of a sample for which $n(t_o + \tau)$ is 47% of n_o.

   ***Solution**:* We are assuming in this problem that

$$\frac{n(t_o + \tau)}{n_o} = 0.47. \tag{12}$$

   Combining (12) and (11) yields the equation

$$0.47 = \exp\left(-\frac{\ln 2}{\tau_{1/2}}\tau\right),$$

   which can be solved for τ to yield

$$\tau = -\frac{\ln(0.47)}{\ln 2}\tau_{1/2}. \tag{13}$$

   Substituting $\tau_{1/2} = 5700$ into (17) yields the estimate

$$\tau \doteq 6209 \text{ years.}$$

□

4. (This problem is based on Problem 13 on page 199 in *Essential Calculus with Applications* by Richard A. Silverman). Suppose that heartwood from a giant sequoia tree has only 75% of the carbon–14 radioactivity of the younger outer wood. Estimate the age of the tree.

   Solution: Since, according to equation (1), radioactivity (the number of disintegrations per unit time) is proportional to the number of radioactive isotopes present in the sample at a particular instant of time, it follows that, in this problem

$$\frac{n(t_o+\tau)}{n_o} = 0.75. \tag{14}$$

   Combining (14) and (11) yields the equation

$$0.75 = \exp\left(-\frac{\ln 2}{\tau_{1/2}}\tau\right),$$

   which can be solved for τ to yield

$$\tau = -\frac{\ln(0.75)}{\ln 2}\tau_{1/2} \doteq 2,364 \text{ years}; \tag{15}$$

   so that the estimated age of the tree is about $2,364$ years. □

5. (This problem is Exercise 8 on page 19 of *Differential Equations and their Applications,* Martin Braun, Fourth Edition, Springer–Verlag, 1993). In the 1950 excavation at Nippur, a city of Babylonia, charcoal from a roof beam gave a count of 4.09 disintegrations per minute per gram. Living wood gave a count of 6.68 disintegrations per minute per gram. Assuming that the charcoal was formed during the time of Hammurabi's reign, find an estimate for the likely time of Hammurabi's succession.

   Solution: We proceed as in Problem 4. In this case, if τ denotes the number of years from Hammurabi's succession to 1950, we have that

$$\frac{n(t_o+\tau)}{n_o} = \frac{4.09}{6.68} \doteq 0.61. \tag{16}$$

   Combining (16) and (11) yields the equation

$$0.61 = \exp\left(-\frac{\ln 2}{\tau_{1/2}}\tau\right),$$

   which can be solved for τ to yield

$$\tau = -\frac{\ln(0.61)}{\ln 2}\tau_{1/2} \doteq 4,065 \text{ years}; \tag{17}$$

   so that a likely estimate for Hammurabi's succession is $4,065$ prior to 1950, or around 2,115 BC. □