Solutions to Assignment #11

1. Use the method of separation of variables to find all solutions of the differential equation
$$\frac{dy}{dt} = te^y.$$

***Solution**:* Separate variables to obtain
$$\int e^{-y}\ du = \int t\ dt. \tag{1}$$
Evaluating the integrals in (1) yields
$$-e^{-y} = \frac{t^2}{2} + c_1, \tag{2}$$
for an arbitrary constant c_1. Multiplying the equation in (2) by -1 and relabeling the constant $-c_1$ by c, we obtain
$$e^{-y} = c - \frac{t^2}{2} \tag{3}$$
Taking the natural logarithm on both sides of (3) yields
$$-y = \ln\left(c - \frac{t^2}{2}\right),$$
from which we get that
$$y(t) = \ln\left(c - \frac{t^2}{2}\right)^{-1}.$$
□

2. Use separation of variables to find all solutions of the differential equation
$$\frac{dy}{dt} = 3ty - t.$$

***Solution**:* Separate variables to obtain
$$\int \frac{1}{y - 1/3}\ dy = \int 3t\ dt,$$

which integrates to

$$\ln\left|y - \frac{1}{3}\right| = \frac{3}{2}t^2 + c_1 \tag{4}$$

Applying the exponential function to both sides of (4) yields

$$\left|y - \frac{1}{3}\right| = c_2 e^{3t^2/2}, \tag{5}$$

where we have set $c_2 = e^{c_1}$. Using the continuity of the exponential function and that of y, we obtain from (5) that

$$y(t) = \frac{1}{3} + ce^{3t^2/2}.$$

□

3. Find a solution of the differential equation

$$\frac{dy}{dt} = y^2$$

satisfying $y = 1$ when $t = 1$. Give the domain of the definition for the function.

***Solution**:* Separating variables we obtain that

$$\int \frac{1}{y^2}\, dy = \int dt,$$

which integrates to

$$-\frac{1}{y} = t + c_1, \tag{6}$$

for arbitrary constant c_1. Multiplying both sides of the equation in (6) by -1 and setting $c = -c_1$, we obtain from (6) that

$$\frac{1}{y} = c - t. \tag{7}$$

Solving for y in (7) yields the general solution

$$y(t) = \frac{1}{c - t}. \tag{8}$$

Next, use the initial condition $y(1) = 1$ to obtain from (8) that

$$\frac{1}{c - 1} = 1,$$

from which we get that $c = 2$, so that

$$y(t) = \frac{1}{2-t}, \qquad \text{for } t < 2,$$

is a solution to the initial value problem

$$\begin{cases} \dfrac{dy}{dt} &= y^2; \\ \\ y(1) &= 1. \end{cases}$$

Observe that the solution is defined on the open interval $(-\infty, 2)$. □

4. Use separation of variable to find a solution of the differential equation

$$\frac{dy}{dt} = \sqrt{y}$$

satisfying $y = 0$ when $t = 0$. Can you come up with another solution of the initial value problem?

***Solution**:* Separating variables we obtain that

$$\int \frac{1}{\sqrt{y}}\, dy = \int dt,$$

which integrates to

$$2\sqrt{y} = t + c, \tag{9}$$

for arbitrary constant c. Dividing the equation in (9) by 2 and squaring on both sides yields

$$y(t) = \frac{1}{4}(t+c)^2 \tag{10}$$

Using the condition $y(0) = 0$ in (10) yields $c = 0$, so that

$$y(t) = \frac{1}{4}t^2, \qquad \text{for all } t \in \mathbb{R},$$

is a solution of the initial value problem

$$\begin{cases} \dfrac{dy}{dt} &= \sqrt{y}; \\ \\ y(0) &= 0. \end{cases} \tag{11}$$

Note that the constant function $v(t) = 0$, for all $t \in \mathbb{R}$, is also a solution of the initial value problem in (11); so the initial value problem in (11) has at least to solutions. □

5. Solve the initial value problem

$$y\frac{dy}{dt} = t, \qquad y(0) = 1.$$

***Solution**:* Separate variable to obtain

$$\int y\ dy = \int t\ dt,$$

which integrates to

$$\frac{1}{2}y^2 = \frac{1}{2}t^2 + c_1, \tag{12}$$

for arbitrary constant c_1. Multiply the equation in (12) by 2 and set $c = 2c_1$ to obtain

$$y^2 = t^2 + c. \tag{13}$$

Using the initial condition, $y(0) = 1$ in (13) yields $c = 1$, so that

$$y(t) = \sqrt{1 + t^2}, \quad \text{for } t \in \mathbb{R},$$

is a solution of the initial value problem. □