Solutions to Assignment #12

1. Solve the initial value problem

$$\frac{dy}{dt} = -y + t, \qquad y(0) = 0. \tag{1}$$

***Solution**:* Rewrite the equation as

$$\frac{dy}{dt} + y = t$$

and multiply by e^t to obtain

$$e^t \frac{dy}{dt} + e^t y = te^t,$$

which can be written as

$$\frac{d}{dt}[e^t y] = te^t, \tag{2}$$

by virtue of the product rule. Integrating on both sides of (2) yields

$$e^t y = \int te^t \, dt. \tag{3}$$

In order to evaluate the integral on the right–hand side of (3), use integration by parts: Set

$$\begin{array}{rcl} u = t & \text{and} & dv = e^t \, dt \\ \text{then, } du = dt & \text{and} & v = e^t, \end{array}$$

so that

$$\int te^t \, dt = te^t - \int e^t \, dt,$$

from which we get that

$$\int te^t \, dt = te^t - e^t + c, \tag{4}$$

where c is an arbitrary constant. Substituting the result in (4) into the right–hand side of (3) yields

$$e^t y = te^t - e^t + c. \tag{5}$$

Solving for y in (6) we obtain

$$y(t) = t - 1 + c\, e^{-t}, \qquad \text{for all } t \in \mathbb{R}. \tag{6}$$

Using the initial condition, $y(0) = 0$, in (6) we have that $-1 + c = 0$, which implies that $c = 1$. Thus,

$$y(t) = t - 1 + e^{-t}, \quad \text{for all } t \in \mathbb{R},$$

solves the initial value problem in (1). □

2. For each $b > 0$, evaluate
$$F(b) = \int_0^b te^{-t}\, dt. \tag{7}$$
Then, compute $\lim\limits_{b\to\infty} F(b)$, if it exists.

***Solution**:* Use integration by parts to evaluate the integral $\int te^{-t}\, dt$. Set

$$\begin{aligned} u = t \quad &\text{and} \quad dv = e^{-t}\, dt \\ \text{then, } du = dt \quad &\text{and} \quad v = -e^{t}, \end{aligned}$$

so that
$$\int te^{-t}\, dt = -te^{-t} + \int e^{-t}\, dt,$$
from which we get that
$$\int te^{-t}\, dt = -te^{-t} - e^{-t} + c, \tag{8}$$
for arbitrary c. We can use the result in (8) to evaluate the definite integral in (7) to obtain that
$$F(b) = \left[-te^{-t} - e^{-t}\right]_0^b = 1 - be^{-b} - e^{-b}. \tag{9}$$
It follows from (9) and L'Hospital's rule that
$$\lim_{b\to\infty} F(b) = 1.$$
□

3. Let $f(t) = t\sin t$ and evaluate the area the region in the ty–plane under the graph of $y = f(t)$, bounded by the t–axis, and between $t = 0$ and $t = \pi$.

Solution: The area of the region is given by $\int_0^\pi t\sin t\ dt$. We evaluate the indefinite integral $\int t\sin t\ dt$ by means of integration by parts. Set

$$\begin{array}{rcl} u = t & \text{and} & dv = \sin t\ dt \\ \text{then,}\ \ du = dt & \text{and} & v = -\cos t, \end{array}$$

so that

$$\int te^{-t}\ dt = -t\cos t + \int \cos t\ dt,$$

from which we get that

$$\int t\sin t\ dt = -t\cos t + \sin t + c, \tag{10}$$

for arbitrary c. We can use the result in (10) to evaluate the definite integral

$$\int_0^\pi t\sin t\ dt = [-t\cos t + \sin t]_0^\pi = \pi.$$

Thus, the area of the region is π. □

4. Let $f(t) = t\ln t$ for all $t > 0$. In Problem 3 of Assignment #5, you were asked to sketch the graph of $y = f(t)$. Evaluate the area of the region in the ty–plane which is below the t–axis and above the graph of $y = f(t)$.

Solution: A sketch of the region, R, is shown in Figure 2. From the sketch of

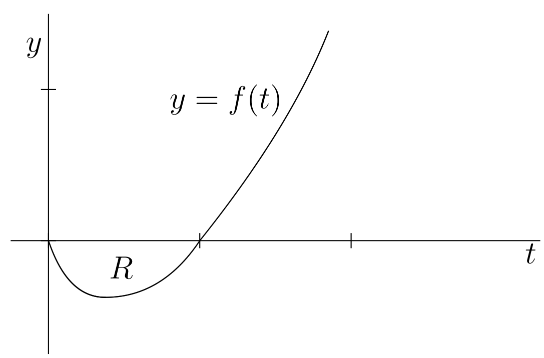

Figure 1: Sketch of graph of $y = f(t)$ in Problem 4

the region we see that the area of the region is

$$\text{area}(R) = -\int_0^1 t\ln t\, dt. \tag{11}$$

In order to evaluate the integral in (11), we first evaluate the indefinite integral $\displaystyle\int t\ln t\, dt$ via integration by parts. Set

$$u = \ln t \quad \text{and} \quad dv = t\, dt$$
$$\text{then,}\quad du = \frac{1}{t}\, dt \quad \text{and} \quad v = \frac{1}{2}t^2,$$

so that

$$\begin{aligned}\int t\ln t\, dt &= \frac{t^2}{2}\ln t - \int \frac{1}{2}t^2 \cdot \frac{1}{t}\, dt \\ &= \frac{t^2}{2}\ln t - \frac{1}{2}\int t\, dt,\end{aligned}$$

so that

$$\int t\ln t\, dt = \frac{t^2}{2}\ln t - \frac{t^2}{4} + c, \tag{12}$$

for arbitrary c.

In order to evaluate the definite integral on the right–hand side of (11), we first evaluate the integral $\displaystyle\int_\varepsilon^1 t\ln t$, for $0 < \varepsilon < 1$. Using (12) we compute

$$\int_\varepsilon^1 t\ln t\, dt = \left[\frac{t^2}{2}\ln t - \frac{t^2}{4}\right]_\varepsilon^1,$$

which yields

$$\int_\varepsilon^1 t\ln t\, dt = -\frac{1}{4} - \frac{\varepsilon^2}{2}\ln\varepsilon + \frac{\varepsilon^2}{4} \tag{13}$$

Observe that, by L'Hospital's rule,

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\frac{\varepsilon^2}{2}\ln\varepsilon &= \lim_{\varepsilon\to 0^+}\frac{\ln\varepsilon}{2/\varepsilon^2} \\ &= \lim_{\varepsilon\to 0^+}\frac{1/\varepsilon}{-4/\varepsilon^3} \\ &= -\frac{1}{4}\lim_{\varepsilon\to 0^+}\varepsilon^2 \\ &= 0.\end{aligned}$$

It then follows from (13) that

$$\int_0^1 t\ln t\; dt = \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 t\ln t\; dt = -\frac{1}{4}. \tag{14}$$

Combining (11) and (14) we see that $\text{area}(R) = \dfrac{1}{4}$, so that the area of the region R in Figure 2 is 1/4. □

5. For each $t > 0$, define $F(t)$ to be the area in the τy–pane under the graph of $y = \tau^2 e^{-\tau}$ from $\tau = 0$ to $\tau = t$.

   (a) Obtain a formula for computing $F(t)$, for $t > 0$.
   ***Solution**:* We begin with

$$F(t) = \int_0^t \tau^2 e^{-\tau}\; d\tau. \tag{15}$$

In order to compute the integral in (15), we first evaluate the indefinite integral $\displaystyle\int \tau^2 e^{-\tau}\; d\tau$ via integration by parts. Set

$$\begin{array}{rcl} u = \tau^2 & \text{and} & dv = e^{-\tau}\; d\tau \\ \text{then,}\;\; du = 2\tau\; d\tau & \text{and} & v = -e^{-\tau}, \end{array}$$

so that

$$\int \tau^2 e^{-\tau}\; d\tau = -\tau^2 e^{-\tau} + \int 2\tau e^{-\tau}\; d\tau. \tag{16}$$

We integrate by parts the integral on the right–hand side of (16) by setting

$$\begin{array}{rcl} u = 2\tau & \text{and} & dv = e^{-\tau}\; d\tau \\ \text{then,}\;\; du = 2\; d\tau & \text{and} & v = -e^{-\tau}, \end{array}$$

so that

$$\int \tau^2 e^{-\tau}\; d\tau = -\tau^2 e^{-\tau} - 2\tau e^{-\tau} + 2\int e^{-\tau}\; d\tau,$$

from which we get that

$$\int \tau^2 e^{-\tau}\; d\tau = -\tau^2 e^{-\tau} - 2\tau e^{-\tau} - 2e^{-\tau} + c, \tag{17}$$

for arbitrary c. Using the result in (17) we obtain from (15) that

$$F(t) = \left[-\tau^2 e^{-\tau} - 2\tau e^{-\tau} - 2e^{-\tau}\right]_0^t,$$

which yields the formula

$$F(t) = 2 - t^2e^{-t} - 2be^{-t} - 2e^{-t} \tag{18}$$

for computing $F(t)$, for $t > 0$. □

(b) Determine the values of t for which $F(t)$ increases or decreases, and the values of t for which the graph of $y = F(t)$ is concave up or concave down.

***Solution**:* We compute $F'(t)$ from (15) via the Fundamental Theorem of Calculus to obtain

$$F'(t) = t^2e^{-t}, \quad \text{for all } t. \tag{19}$$

Differentiate (19) with respect to t to obtain

$$F''(t) = 2te^{-t} - t^2e^{-t}, \quad \text{for all } t, \tag{20}$$

where we have used the product rule and the Chain Rule. Simplifying the right–hand side of (20) yields

$$F''(t) = t(2-t)e^{-t}, \quad \text{for all } t. \tag{21}$$

It follows from (19) the $F'(t) > 0$ for all $t > 0$; thus, $F(t)$ increases for all $t > 0$.

Next, obtain from (21) that, for $t > 0$, $F''(t) > 0$ for $0 < t < 2$, and $F''(t) < 0$ for $t > 2$. We therefore conclude that the graph of $y = F(t)$ concave up over the interval $(0, 2)$, and concave down for $t > 2$. □

(c) Sketch the graph of $y = F(t)$.

***Solution**:* First, note that from the formula in we obtain that

$$\lim_{t\to\infty} F(t) = 2,$$

where we have used L'Hospital's Rule. Using this information along with the quialitative information obtained in part (b) we get the gaph of $y = F(t)$, for $t > 0$, sketched in Figure 2. □

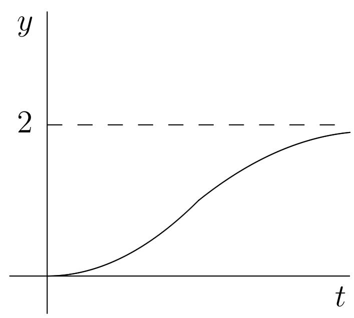

Figure 2: Sketch of graph of $y = F(t)$