Solutions to Assignment #14

1. For any population, ignoring migration, harvesting, or predation, one can model the relative growth rate by the following conservation principle

$$\frac{1}{N}\frac{dN}{dt} = \text{birth rate (per capita)} - \text{death rate (per capita)} = b - d,$$

where b and d could be functions of time and the population density N.

(a) Suppose that b and d are linear functions of N given by $b = b_o - \alpha N$ and $d = d_o + \beta N$ where b_o, d_o, α and β are positive constants. Assume that $b_o > d_o$. Sketch the graphs of b and d as functions of N. Give a possible interpretation for these graphs.

***Solution**:* Sketches of the graphs of the functions $b = b(N)$ and $d = d(N)$ are shown in Figure 1. In this model we assume that the *per–capita* birth

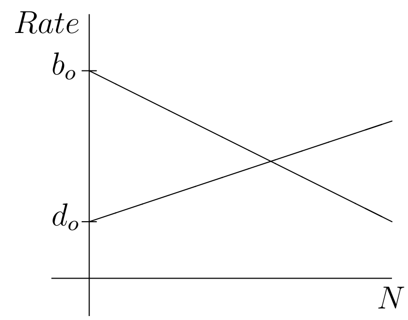

Figure 1: Sketch of graph birth and death rates

rate is a linear function of the population density that decreases with increasing density; while the *per–capita* death rate increases linearly with increasing density. □

(b) Find the point where the two lines sketched in part (a) intersect. Let K denote the first coordinate of the point of intersection. Show that $K = \dfrac{b_o - d_o}{\alpha + \beta}$. K is the carrying capacity of the population.

***Solution**:* The two lines in Figure 1 intersect when

$$b_o - \alpha N = d_o + \beta N,$$

which yields

$$N = \frac{b_o - d_o}{\alpha + \beta};$$

this is the carrying capacity. □

(c) Show that $\dfrac{dN}{dt} = rN\left(1 - \dfrac{N}{K}\right)$ where $r = b_o - d_o$ is the intrinsic growth rate.

Solution: Substitutes the expressions for b and d into the model

$$\frac{1}{N}\frac{dN}{dt} = b - d$$

to get

$$\begin{aligned} \frac{1}{N}\frac{dN}{dt} &= b_o - \alpha N - d_o - \beta N \\ &= b_o - d_o - (\alpha+\beta)N \\ &= (b_o - d_o)\left[1 - \frac{\alpha+\beta}{b_o - d_o}N\right] \\ &= (b_o - d_o)\left[1 - \frac{1}{K}\,N\right], \end{aligned}$$

where we have used $K = \dfrac{b_o - d_o}{\alpha + \beta}$. Setting $r = b_o - d_o$, we obtain the Logistic equation $\dfrac{dN}{dt} = rN\left(1 - \dfrac{N}{K}\right)$. □

2. Assume that a population of size $N = N(t)$ grows according to a logistic model with carrying capacity of 5×10^8 individuals. Assume also that, when the population size is very small, the population doubles every 30 minutes. Suppose the initial population is 10^8. Estimate the size of the population two hours later.

Solution: For $N_o > 0$, the solution to the initial value problem

$$\begin{cases} \dfrac{dN}{dt} = rN\left(1 - \dfrac{N}{K}\right); \\ \\ N(0) = N_o, \end{cases} \tag{1}$$

is given by

$$N(t) = \frac{N_o K}{N_o + (K - N_o)e^{-rt}}, \qquad \text{for } t > 0. \tag{2}$$

In this problem we have that $K = 5 \times 10^8$ and $N_o = 10^8$. We need to estimate the intrinsic growth rate r. For small population density, r can be approximated by the Malthusian growth rate,

$$\frac{\ln 2}{\tau_2},$$

where τ_2 is the doubling time. Thus,

$$r \doteq \frac{\ln 2}{0.5} = 2\ln 2 \doteq 1.3863, \tag{3}$$

where we have used the doubling time in hours.

Using the expression for $N(t)$ in (1) we see that the population size two hours later will be

$$N(2) = \frac{N_o K}{N_o + (K - N_o)e^{-2r}}. \tag{4}$$

Substituting the given values for N_o and K, and the estimate for r in (3), we obtain from (4) the estimate

$$N(2) \doteq 4 \times 10^8.$$

□

3. Let $N = N(t)$ denote the size of the population described in Problem 2, where t is measured in hours. Estimate the time that it will take the population to grow to 90% of its carrying capacity.

***Solution**:* We solve the equation

$$N(t) = 0.90K,$$

for t, where $N(t)$ is given by (2) to get

$$\frac{N_o K}{N_o + (K - N_o)e^{-rt}} = 0.90K,$$

or

$$\frac{N_o}{N_o + (K - N_o)e^{-rt}} = 0.90. \tag{5}$$

Rearranging the equation in (5) yields

$$0.10N_o = (K - N_o)e^{-rt},$$

which simplifies to

$$0.10 = 4e^{-rt},$$

or

$$e^{rt} = 40,$$

so that

$$t = \frac{1}{r}\ln(40). \tag{6}$$

Using the estimate for r in (3) obtained in Problem 2, we obtain from (6) that

$$t \doteq 2.66 \text{ hours}.$$

Thus, it will take about 2 hours and 40 minutes for the population to grow to 90% of its carrying capacity. □

4. Suppose that a population of size $N = N(t)$ grows according to the Logistic model. Assume that the population grows from a size N_1 to a size N_2 is an interval of time of length T. Show that

$$T = \int_{N_1}^{N_2} \frac{K}{rN(K-N)}\, dN, \tag{7}$$

where K is the carrying capacity and r is the intrinsic growth rate.

***Solution**:* Set $N_1 = N(t_1)$ and $N_2 = N(t_2)$ and $T = t_2 - t_1$. We want to find a formula for T.

From the Logistic equation we obtain

$$\frac{dt}{dN} = \frac{K}{rN(K-N)},$$

which can be separated to yield

$$dt = \frac{K}{rN(K-N)}\, dN. \tag{8}$$

Next, integrate on the left–hand side of (8) from t_1 to t_2 and on the right–hand side from $N(t_1)$ to $N(t_2)$ to get

$$\int_{t_1}^{t_2} dt = \int_{N_1}^{N_2} \frac{K}{rN(K-N)}\, dN,$$

which yields (7) since $T = t_2 - t_1$. □

5. Suppose a population of size $N = N(t)$ grows logistically with intrinsic growth rate r and carrying capacity K. Use the formula (7) derived in Problem 4 to answer the following questions.

   (a) Calculate the time that it takes for the population size to grow from $N_1 = K/4$ to $N_2 = K/2$.

   ***Solution**:* Rewrite the formula for T in (7) as

$$T = -\frac{K}{r}\int_{N_1}^{N_2} \frac{1}{N(N-K)}\, dN, \tag{9}$$

and use partial fractions to evaluate the integral on the right–hand side of (9). Write

$$\frac{1}{N(N-K)} = \frac{A}{N} + \frac{B}{N-K}, \tag{10}$$

where the constants A and B need to be determined. In then follows from (10) that

$$\int \frac{1}{N(N-K)}\, dN = A\ln|N| + B\ln|N-K| + c, \tag{11}$$

for arbitrary constant c.

In order to find the constants A and B, first multiply both sides of the equation in (10) by $N(N-K)$ to get

$$1 = A(N-K) + BN,$$

or

$$0N + 1 = (A+B)N - AK. \tag{12}$$

Note that the right–hand side of the equation in (12) is polynomial in N. The constant, 1, in the left–hand side of (12) can also be thought of as a polynomial in N when written as $0N + 1$. Two polynomials are equal if and only if corresponding coefficients are equal. Hence, the equality in (12) implies that

$$\left\{\begin{array}{rcl} A+B & = & 0 \\ -AK & = & 1. \end{array}\right. \tag{13}$$

solving the second equation in (13) for A yields

$$A = -\frac{1}{K}. \tag{14}$$

Substituting the value for A in (14) into the first equation in (13) and solving for B yields

$$B = \frac{1}{K}. \tag{15}$$

Substituting the values for A and B in (14) and (15), respectively, into (11) yields the integration formula

$$\int \frac{1}{N(N-K)}\, dN = -\frac{1}{K}\ln|N| + \frac{1}{K}\ln|N-K| + c, \tag{16}$$

for arbitrary constant c. We can then use (16) to obtain from (9)

$$\begin{aligned} T &= -\frac{K}{r}\left[-\frac{1}{K}\ln|N| + \frac{1}{K}\ln|N-K|\right]_{N_1}^{N_2} \\ &= \frac{1}{r}\left[\ln|N| - \ln|N-K|\right]_{N_1}^{N_2} \\ &= \frac{1}{r}\left[\ln\frac{|N|}{|N-K|}\right]_{N_1}^{N_2}, \end{aligned} \tag{17}$$

where we have used the properties of the natural logarithm function in the last step in (17). It follows from the calculations in (17) that

$$T = \frac{1}{r}\left[\ln\frac{|N_2|}{|N_2-K|} - \ln\frac{|N_1|}{|N_1-K|}\right]. \tag{18}$$

For the case in which $N_1 = K/4$ and $N_2 = K/2$, we obtain from (18) that

$$T = \frac{1}{r}\left[\ln\frac{K/2}{K/2} - \ln\frac{K/4}{3K/4}\right] = \frac{\ln 3}{r}.$$

□

(b) What happens to T in (7) as N_2 tends to K?

***Solution**:* Use the formula for T in (18) to compute

$$\lim_{N_2\to K} T = \lim_{N_2\to K}\frac{1}{r}\left[\ln\frac{|N_2|}{|N_2-K|} - \ln\frac{|N_1|}{|N_1-K|}\right] = +\infty,$$

since $|N_2 - K| \to 0$ as $N_2 \to K$. Thus, T tends to infinity as $N_2 \to K$. □