Assignment #17

Due on Wednesday, November 23, 2016

Read Chapter 5, *Applications of Differentiable Calculus, Part II,* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`, starting on page 75.

Read Section 5.1, *Linear Approximations,* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`, starting on page 76.

Background and Definitions

Let $f\colon I \to \mathbb{R}$ denote a differentiable function defined on some open interval I, which contains a. The linear approximation to f around a is defined by

$$L(x;a) = f(a) + f'(a)(x-a), \qquad \text{for all } x \in \mathbb{R}.$$

The linear function L approximates f around a in the sense that

$$f(x) = L(x;a) + E(a,x),$$

where the error term, E, satisfies

$$\lim_{x\to a} \frac{|E(a;x)|}{|x-a|} = 0.$$

If f is twice differentiable, the error terms is given by

$$E(a;x) = f(x) - L(x;a) = \int_a^x f''(t)(x-t)\ \mathrm{d}t.$$

Hence, if $|f''(x)| \leq M$ for some constant M in some interval around a, then

$$|E(x;a)| \leq \frac{M}{2}|x-a|^2$$

for x in that interval.

Do the following problems.

1. Let $f(x) = \dfrac{1}{\sqrt{1+x}}$ for $x > -1$. Give the linear approximation to f around $a = 0$.

2. Let $f(x) = e^{-x}$ for all $x \in \mathbb{R}$. Give the linear approximation to f around $a = 1$.

3. Let $f \colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sin(x)$ for all $x \in \mathbb{R}$.

   (a) Give the linear approximation for $f(x)$ near $a = \pi/6$.

   (b) Estimate the error term $E(x; \pi/6) = \displaystyle\int_{\pi/6}^{x} f''(t)(x - t)\ \mathrm{d}t$.

   (c) How far can x be from $\pi/6$ so that the approximation is good to two decimal places?

   (d) Estimate $\sin(0.51)$. Compare with the approximation obtained with a calculator.

4. Let $f \colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = e^{-x}$ for all $x \in \mathbb{R}$.

   (a) Give the linear approximation for $f(x)$ near $a = 0$.

   (b) Estimate the error term $E(x; 0) = \displaystyle\int_{0}^{x} f''(t)(x - t)\ \mathrm{d}t$ for $x > 0$, using the estimate $e^{-x} \leq 1$ for all $x \geq 0$.

   (c) How far can $x > 0$ be from 0 so that the approximation is good to two decimal places?

   (d) Estimate $1/e^{0.09}$. How accurate is your estimate?

5. *Linear Approximations*[1]. Multiply the linear approximation to e^x near $a = 0$ by itself to obtain an approximation for e^{2x}. Compare this with the linear approximation you obtain for the function $f \colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^{2x}$ for all $x \in \mathbb{R}$. Explain why the two approximations to e^{2x} are consistent, and discuss which one is more accurate.

[1] Adapted from Problem 8 on page 153 in Hughes–Hallett et al, *Calculus,* Third Edition, Wiley, 2002