Solutions to Assignment #17

1. Let $f(x) = \dfrac{1}{\sqrt{1+x}}$ for $x > -1$. Give the linear approximation to f around $a = 0$.

   ***Solution**:* Compute
   $$L(x;0) = f(0) + f'(0)x,$$
   where
   $$f'(x) = -\frac{1}{2(1+x)^{3/2}}, \qquad \text{for } x > -1.$$
   Thus,
   $$L(x;0) = 1 - \frac{1}{2}x, \qquad \text{for } x \in \mathbb{R}.$$
   □

2. Let $f(x) = e^{-x}$ for all $x \in \mathbb{R}$. Give the linear approximation to f around $a = 1$.

   ***Solution**:* Compute
   $$L(x;1) = f(1) + f'(1)(x-1),$$
   where
   $$f'(x) = -e^{-x}, \qquad \text{for } x \in \mathbb{R}.$$
   Thus,
   $$L(x;1) = e^{-1} - e^{-1}(x-1), \qquad \text{for } x \in \mathbb{R},$$
   or
   $$L(x;1) = e^{-1}(2-x), \qquad \text{for } x \in \mathbb{R}.$$
   □

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sin(x)$ for all $x \in \mathbb{R}$.

   (a) Give the linear approximation for $f(x)$ near $a = \pi/6$.
   ***Solution**:* Compute
   $$L(x;\pi/6) = f(\pi/6) + f'(\pi/6)\left(x - \frac{\pi}{6}\right), \qquad \text{for } x \in \mathbb{R},$$
   where
   $$f'(x) = \cos x, \qquad \text{for } x \in \mathbb{R}.$$

Thus,

$$L(x;\pi/6) = \sin(\pi/6) + \cos(\pi/6)\left(x - \frac{\pi}{6}\right), \qquad \text{for } x \in \mathbb{R},$$

or

$$L(x;\pi/6) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right), \qquad \text{for } x \in \mathbb{R}. \tag{1}$$

□

(b) Estimate the error term $E(x;\pi/6) = \displaystyle\int_{\pi/6}^{x} f''(t)(x-t)\mathrm{d}t$.

***Solution**:* Use the estimate

$$|E(x;\pi/6)| \leqslant \frac{M}{2}\left|x - \frac{\pi}{6}\right|^2, \qquad \text{for } x \in \mathbb{R},$$

where $M = 1$, since

$$|\sin''(t)| = |\sin(t)| \leqslant 1, \qquad \text{for } t \in \mathbb{R}.$$

Thus,

$$|E(x;\pi/6)| \leqslant \frac{1}{2}\left|x - \frac{\pi}{6}\right|^2, \qquad \text{for } x \in \mathbb{R}. \tag{2}$$

□

(c) How far can x be from $\pi/6$ so that the approximation is good to two decimal places?

***Solution**:* We want

$$|E(x,\pi/6)| < 0.005;$$

so that, in view of (2),

$$\frac{1}{2}\left|x - \frac{\pi}{6}\right|^2 < 0.005,$$

from which we get

$$\left|x - \frac{\pi}{6}\right| < 0.1.$$

Thus, if x is within 0.1 of $\pi/6$, the linear approximation to $\sin x$ will be accurate to at least two decimal places. □

(d) Estimate $\sin(0.51)$. Compare with the approximation obtained with a calculator.

***Solution**:* Note that $\pi/6$ is about 0.5236. Thus, using the linear approximation to $\sin x$ at $a = \pi/6$ in (1), we get that

$$\sin(0.51) \approx L(0.51;\pi/6) = \frac{1}{2} + \frac{\sqrt{3}}{2}(0.51 - 0.5236),$$

so that

$$\sin(0.51) \approx 0.4882 \tag{3}$$

Using the estimate for the error in the approximation in (2) we see that

$$|E(0.51; \pi/6)| \leqslant \frac{1}{2}|0.51 - 0.5236|^2 \doteq 0.0001.$$

Thus, the estimate in (3) is accurate to three decimal places, so that

$$\sin(0.51) \doteq 0.488.$$

The estimate given by a calculator is

$$\sin(0.51) \doteq 0.48817724688290749450013023767457,$$

which agrees with the estimate in (3) to four decimal places after rounding up. □

4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = e^{-x}$ for all $x \in \mathbb{R}$.

   (a) Give the linear approximation for $f(x)$ near $a = 0$.
   ***Solution**:* Compute

   $$L(x; 0) = f(0) + f'(0)x,$$

   where

   $$f'(x) = -e^{-x}, \quad \text{for } x \in \mathbb{R}.$$

   Thus,

   $$L(x; 0) = 1 - x, \quad \text{for } x \in \mathbb{R}. \tag{4}$$

   □

   (b) Estimate the error term $E(x; 0) = \displaystyle\int_0^x f''(t)(x - t)\mathrm{d}t$ for $x > 0$, using the estimate $e^{-x} \leq 1$ for all $x \geq 0$.
   ***Solution**:* We use the estimate

   $$|E(x; 0)| \leqslant \frac{M}{2}\,|x|^2\,, \quad \text{for } x \geqslant 0,$$

   where $M = 1$, since

   $$|e^{-t}| \leqslant 1, \quad \text{for } t \geqslant 0.$$

   Thus,

   $$|E(x; 0)| \leqslant \frac{1}{2}\,|x|^2\,, \quad \text{for } x \geqslant 0. \tag{5}$$

   □

(c) How far can $x > 0$ be from 0 so that the approximation is good to two decimal places?

***Solution**:* For two decimal places of accuracy, in view of (5), we want

$$\frac{1}{2}x^2 < 0.005,$$

so that

$$0 \leqslant x < 0.1.$$

So, if $x > 0$ is within 0.1 of 0, then the linear approximation in (4) should yield an estimate to e^{-x} which is accurate to two decimal paces. □

(d) Estimate $1/e^{0.09}$. How accurate is your estimate?

***Solution**:* Write $1/e^{0.09} = e^{-0.09}$. Thus, using the linear approximation to e^{-x} in (4), we approximate

$$e^{-0.09} \approx 1 - 0.09 = 0.91. \tag{6}$$

Using the error estimate in (5) we see that the error in the approximation in (6) is at most

$$|E(0.09; 0)| \leqslant \frac{1}{2}\,|0.09|^2 \approx 0.00405;$$

thus, the estimate in (6) is accurate to at least two decimal places. □

5. *Linear Approximations*[1]. Multiply the linear approximation to e^x near $a = 0$ by itself to obtain an approximation for e^{2x}. Compare this with the linear approximation you obtain for the function f $f(x) = e^{2x}$ for all $x \in \mathbb{R}$. Explain why the two approximations to e^{2x} are consistent, and discuss which one is more accurate.

***Solution**:* Let $f(x) = e^x$ for all $x \in \mathbb{R}$. We first compute the linear to f near $a = 0$:

$$L(x; 0) = f(0) + f'(0)x, \quad \text{for all } x \in \mathbb{R},$$

where

$$f'(x) = e^x, \quad \text{for all } x \in \mathbb{R}.$$

Thus,

$$L(x; 0) = 1 + x, \quad \text{for } x \in \mathbb{R}. \tag{7}$$

[1] Adapted from Problem 8 on page 153 in Hughes–Hallett et al, *Calculus,* Third Edition, Wiley, 2002

Next, let $g(x) = e^{2x}$ for all $x \in \mathbb{R}$, so that $g(x) = [f(x)]^2$ for all $x \in \mathbb{R}$. Thus,

$$[L(x;0)]^2 = (1+x)^2 = 1 + 2x + x^2, \quad \text{for } x \in \mathbb{R}, \tag{8}$$

is an approximation to g near $a = 0$. On the other hand, the linear approximation to g at $a = 0$ is

$$L_g(x;0) = g(0) + g'(0)x, \quad \text{for all } x \in \mathbb{R},$$

where

$$g'(x) = 2e^{2x}, \quad \text{for all } x \in \mathbb{R},$$

so that,

$$L_g(x;0) = 1 + 2x, \quad \text{for } x \in \mathbb{R}. \tag{9}$$

The approximations $[L(x;0)]^2$ and $L_g(x;0)$ are consistent in the sense that the linear part of $[L(x;0)]^2$ agrees with the linear approximation $L_g(x;0)$.

To see which one of the approximations $[L(x;0)]^2$ and $L_g(x;0)$ is more accurate, observe that the graph of $y = g(x)$ at $x = 0$ is concave up since $g''(0) = 4 > 0$. However, the graph of $y = L_g(x)$ has no concavity at $x = 0$, while that of $y = [L(x;0)]^2$ is also concave up at 0. Hence, $[L(x;0)]^2$ is a more accurate approximation. Figure 1, generated with WolframAlpha®, shows this argument graphically. □

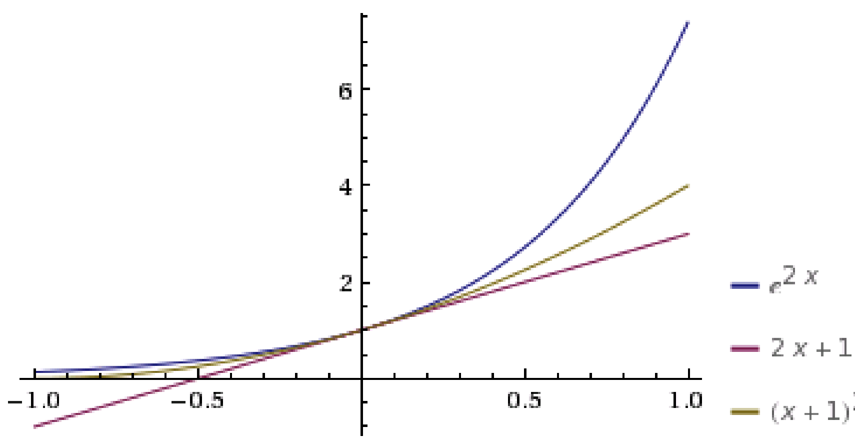

Figure 1: Sketch of the graphs of $y = e^{2x}$, $y = [L(x;0)]^2$ and $y = L_g(x;0)$