Solutions to Assignment #18

1. For the following first–order differential equations, find all the equilibrium solutions and use the principle of linearized stability, when applicable, to determine whether the equilibrium solutions are asymptotically stable or unstable.

   (a) $\dfrac{dy}{dt} = y^2 - y - 6.$

   Solution: Set $f(y) = y^2 - y - 6$ for $y \in \mathbb{R}$, and observe that $f(y) = (y+2)(y-3)$. Thus, $\overline{y}_1 = -2$ and $\overline{y}_2 = 3$ are equilibrium points of the equation. In order to apply the principle of linearized stability, compute $f'(y) = 2y - 1$. Then, since $f'(-2) = -5 < 0$, the equilibrium point $\overline{y}_1 = -2$ is asymptotically stable. Similarly, since $f'(3) = 5 > 0$, the equilibrium point $\overline{y}_2 = 3$ is unstable. □

   (b) $\dfrac{dy}{dt} = 0.5y(y-4)(y+2).$

   Solution: Set $f(y) = 0.5y(y-4)(y+2)$ for $y \in \mathbb{R}$. The equilibrium points of the equation are $\overline{y}_1 = -2$, $\overline{y}_2 = 0$ and $\overline{y}_3 = 4$. The derivative of f is given by

   $$f'(y) = \frac{3}{2}y^2 - 2y - 4.$$

   We then have that $f'(-2) = 6 > 0$; so that, $\overline{y}_1 = -2$ is unstable, by the principle of linearized stability; next, compute $f'(0) = -4 < 0$, so that $\overline{y}_2 = 0$ is asymptotically stable by the principle of linearized stability; finally, $f'(4) = 12 > 0$; so that, $\overline{y}_1 = 4$ is unstable, by the principle of linearized stability. □

2. For the following first–order differential equations, find all the equilibrium solutions and use the principle of linearized stability, when applicable, to determine whether the equilibrium solutions are asymptotically stable or unstable.

   (a) $\dfrac{dy}{dt} = (y-1)^2(y+2).$

   Solution: Set $f(y) = (y-1)^2(y+2)$ for $y \in \mathbb{R}$. The equilibrium points are $\overline{y}_1 = -2$ and $\overline{y}_2 = 1$. The derivative of f is given by

   $$f'(y) = 3(y-1)(y+1).$$

   Compute $f'(-2) = 9 > 0$; thus, $\overline{y}_1 = -2$ is unstable by the principle of linearized stability. Note that $f'(1) = 0$, so the principle of linearized stability does not apply. □

(b) $\dfrac{dy}{dt} = \dfrac{y^2-1}{y^2+1}$.

Solution: Set $f(y) = \dfrac{y^2-1}{y^2+1}$, for $y \in \mathbb{R}$, or $f(y) = \dfrac{(y+1)(y-1)}{y^2+1}$, for $y \in \mathbb{R}$, so that the equilibrium points of the differential equation are $\overline{y}_1 = -1$ and $\overline{y}_2 = 1$.

Next, compute

$$f'(y) = \frac{4y}{(y^2+1)^2}, \qquad \text{for } y \in \mathbb{R}.$$

Then, $f'(-1) < 0$ for that $\overline{y}_1 = -1$ is asymptotically stable by the principle of linearized stability. On the other hand, $f'(1) > 0$, so that $\overline{y}_2 = 1$ is unstable by the principle of linearized stability. □

3. Give the linearization of the equation $\dfrac{dy}{dt} = \sin y$ around the equilibrium point $\overline{y} = \pi$. Solve the linearized equation and determine the long term behavior of its solutions as $t \to \infty$.

Solution: Set $u = y - \pi$. The linearization of the equation is

$$\frac{du}{dt} = \sin'(\pi)u,$$

or

$$\frac{du}{dt} = -u.$$

Thus, the general solution of the linearized equation is

$$u(t) = c\, e^{-t}, \qquad \text{for } t \in \mathbb{R}.$$

This,

$$\lim_{t\to\infty} u(t) = 0.$$

□

4. Determine whether the equilibrium solution $\overline{y} = 0$ is stable or unstable for the following equations.

(a) $\dfrac{dy}{dt} = y^2$ (b) $\dfrac{dy}{dt} = -y^2$ (c) $\dfrac{dy}{dt} = y^3$ (b) $\dfrac{dy}{dt} = -y^3$

Solution: Note that $\overline{y} = 0$ is the only equilibrium point for all the equations and that the principle of linearized stability does not apply since $f'(0) =$ in all cases. Thus, we have to resort to other means to determine the stability of $\overline{y} = 0$.

(a) Note that, for $y_o > 0$, the the solution to the initial value problem

$$\begin{cases} \dfrac{dy}{dt} & = & y^2; \\ y(0) & = & y_o, \end{cases}$$

namely,

$$y(t) = \frac{y_o}{1 - y_o t}, \qquad \text{for } t < \frac{1}{y_o},$$

does not exist for all $t > 0$. Thus, $\overline{y} = 0$ is unstable in this case.

(b) In this case, solve the initial value problem

$$\begin{cases} \dfrac{dy}{dt} & = & -y^2; \\ y(0) & = & y_o, \end{cases} \tag{1}$$

for $y_o < 0$ to obtain

$$y(t) = \frac{y_o}{1 + y_o t}, \qquad \text{for } t < \frac{1}{-y_o}.$$

Thus, the solutions to the initial value problem (1) for $y_o < 0$ does not exist for all $t > 0$. Thus, $\overline{y} = 0$ is unstable in this case as well.

(c) Solve the initial value problem

$$\begin{cases} \dfrac{dy}{dt} & = & y^3; \\ y(0) & = & y_o, \end{cases} \tag{2}$$

for $y_o \neq 0$ to obtain

$$y(t) = \frac{|y_o|}{\sqrt{1 - 2y_o^2 t}}, \qquad \text{for } t < \frac{1}{2y_o^2}.$$

Thus, the solutions to the initial value problem (2) for any $y_o \neq 00$ does not exist for all $t > 0$. Thus, $\overline{y} = 0$ is unstable in this case as well.

(d) Solve the initial value problem

$$\begin{cases} \dfrac{dy}{dt} & = & -y^3; \\ y(0) & = & y_o, \end{cases} \tag{3}$$

for $y_o \neq 0$ to obtain

$$y(t) = \frac{|y_o|}{\sqrt{1 + 2y_o^2 t}}, \qquad \text{for } t > 0.$$

Note that

$$\lim_{t\to\infty} y(t) = 0,$$

so that $\overline{y} = 0$ is asymptotically stable.

□

5. For which values of y_o does the IVP $\begin{cases} \dfrac{dy}{dt} &= y^3 - y \\ y(0) &= y_o \end{cases}$ have a solution, $y(t)$, which exists for all $t > 0$? For those values of y_o, what is $\lim\limits_{t\to\infty} y(t)$?

***Solution**:* Set $f(y) = y^3 - y$ and observe that $f(y) = y(y+1)(y-1)$, so that the equation

$$\frac{dy}{dt} = f(y),$$

has equilibrium points $\overline{y}_1 = -1$, $\overline{y}_2 = 0$ and $\overline{y}_3 = 1$. Thus, if y_o is any of these points, then the initial value problem

$$\begin{cases} \dfrac{dy}{dt} &= y^3 - y; \\ y(0) &= y_o \end{cases} \tag{4}$$

has constant solutions that exist for all t equal to the equilibrium points. Thus, if $y_o = -1$, the solution to the IVP in (4) satisfies

$$\lim_{t\to\infty} y(t) = -1.$$

Similarly, if $y_o = 0$, then

$$\lim_{t\to\infty} y(t) = 0,$$

and, if $y_o = 1$, then

$$\lim_{t\to\infty} y(t) = 1.$$

Next, according to the global existence and long–term behavior theorem, for any $y_o \in \mathbb{R}$ such that $-1 < y_o < 0$ or $0 < y_o < 1$, the the solution to the IVP in (4) exists for all t. The long–term behavior of the solution will depend on the stability properties of the equilibrium points.

In order to determine the stability properties of the equilibrium points, we will apply the principle of linearized stability. Compute $f'(y) = 3y^2 - 1$ so that

$$f'(-1) = 2 > 0,$$

which implies that $\overline{y}_1 = -1$ is unstable; similarly, $\overline{y}_3 = 1$ is also unstable. On the other hand, $f'(0) = -1 < 0$, which shows that $\overline{y}_2 = 0$ is asymptotically stable. We therefore conclude that

$$\lim_{t\to\infty} y(t) = 0, \qquad \text{for } -1 < y_o < 1.$$

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