Solutions to Assignment #2

1. Let $N(t)$ denote the size of a bacterial population in culture at time t. $N(t)$ can be measured by weight (e.g., grams), or by concentration via optical density measurements. Assume that $N = N(t)$ is twice differentiable and that it satisfies the following differential equation:

$$\frac{dN}{dt} = 1.24N - 3.60N^2, \tag{1}$$

where $N = N(t)$ measures the concentration of bacteria obtained via optical density measurements.

Using the information provided by the differential equation in (1) to answer the following questions.

(a) Find the values of N for which the population size is not changing; that is the values of N for which $\frac{dN}{dt} = 0$.

Solution: Write the equation in (1) in the form

$$\frac{dN}{dt} = 3.6\ N\left(\frac{31}{90} - N\right), \tag{2}$$

we see that $\frac{dN}{dt} = 0$ when $N = 0$ and $N = \frac{31}{90}$. □

(b) Find the range of positive values of N for which the population size is increasing; that is the values of N for which $\frac{dN}{dt} > 0$.

Solution: We see from (2) that $\frac{dN}{dt} > 0$ for $0 < N < \frac{31}{90}$; so that $N(t)$ increases for positive values of N less than $\frac{31}{90}$. □

(c) Find the range of positive values of N for which the population size is decreasing; that is the values of N for which $\frac{dN}{dt} < 0$.

Solution: According to (2), $\frac{dN}{dt} < 0$ for $N > \frac{31}{90}$; it then follows from (2) that $N(t)$ decreases for values of N higher than $\frac{31}{90}$. □

2. Use the differential equation in (1) and the Chain Rule to obtain an expression for the second derivative of N with respect to t, $\dfrac{d^2N}{dt^2}$. Put your answer in the form
$$\frac{d^2N}{dt^2} = g(N), \tag{3}$$
where g is a function of a single variable.

   Solution: Differentiate with respect to t on both sides of (1) to obtain
$$\frac{d^2N}{dt^2} = 1.24\frac{dN}{dt} - 7.2\ N\frac{dN}{dt}, \tag{4}$$
where we have used the Chain Rule when taking the derivative of the second term on the right–hand side (1). The right–hand side of the equation in (4) can be factored to yield
$$\frac{d^2N}{dt^2} = 7.2\left(\frac{31}{180} - N\right)\frac{dN}{dt}. \tag{5}$$
Substituting the expression for $\dfrac{dN}{dt}$ in (2) into the right–hand side of the equation in (5) yields
$$\frac{d^2N}{dt^2} = 2(3.6)^2\ N\left(\frac{31}{180} - N\right)\left(\frac{31}{90} - N\right),$$
which can be re–written as
$$\frac{d^2N}{dt^2} = 2(3.6)^2\ N\left(N - \frac{31}{180}\right)\left(N - \frac{31}{90}\right). \tag{6}$$
Set $K = \dfrac{31}{90}$; then, (6) can be written as
$$\frac{d^2N}{dt^2} = 2(3.6)^2\ N\left(N - \frac{K}{2}\right)(N - K). \tag{7}$$
□

3. Based on your answer to Problem 2 in the form of equation (5),

   (a) find the values of N for which the graph of $N = N(t)$ (that is, graph of N as a function of t in the tN–plane), might have an inflection point; that is, find the values of N for which $\dfrac{d^2N}{dt^2} = 0$;

Solution: According to (6), the graph of $N = N(t)$ might have an inflection point a the values

$$N = 0, \quad N = \frac{31}{180}, \quad \text{or} \quad N = \frac{31}{90}.$$

□

(b) find the range of positive values of N for which the graph of $N = N(t)$ is concave up; that is the values of N for which $\frac{d^2N}{dt^2} > 0$;

Solution: Set $K = \frac{31}{90}$; then, according to (7), the sign of the second derivative of N with respect to t, for positive values of N, is determined by the signs of the right–most factors on the right–hand side of (7):

$$N - \frac{K}{2} \quad \text{and} \quad N - K.$$

The signs of these two factors are displayed in Table 1. The con-

	$0 < N < K/2$	$K/2 < N < K$	$N > K$
$N - \frac{K}{2}$	$-$	$+$	$+$
$N - K$	$-$	$-$	$+$
$N''(t)$	$+$	$-$	$+$
graph of $N(t)$	concave–up	concave–down	concave–up

Table 1: Concavity of the graph of $N = N(t)$

cavity of of the graph of $N = N(t)$ is also displayed in Table 1. From that table we get that the graph of $N = N(t)$ is concave up for

$$0 < N < \frac{K}{2} \quad \text{or} \quad N > K.$$

□

(c) find the range of positive values of N for which the graph of $N = N(t)$ is concave down; that is the values of N for which $\frac{d^2N}{dt^2} < 0$.

Solution: According to the results displayed in Table 1, the graph of $N = N(t)$ is concave down for

$$\frac{K}{2} < N < K.$$

□

4. Suppose that $N = N(t)$ is a solution to the differential equation in (1). Use the qualitative information about the graph of $N = N(t)$ obtained in Problems 2 and 3 to sketch possible graphs of N for $N \geqslant 0$.

   Based on your sketches, explain what the population model in (1) seems to be predicting.

   Solution: Possible graphs of $N = N(t)$, for a solution of (1) are sketched in Figure 1. According to the sketches in Figure 1, the

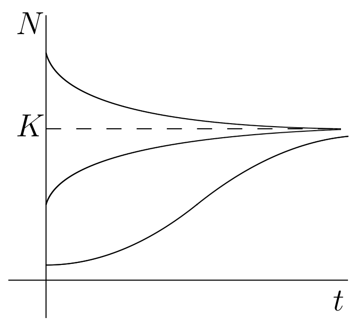

Figure 1: Sketch of graph of $N = N(t)$

logistic model in (1) seems to predict that, for positive population values, the solution will tend towards $K = \dfrac{31}{90}$. □

5. Analysis of certain one–compartment dilution model yields the differential equation

$$\frac{dQ}{dt} = a\left(1 - \frac{Q}{L}\right), \tag{8}$$

   for positive constants a and L.

   Assume that the differential equation in (8) has a solution, $Q = Q(t)$, which is twice–differentiable.

(a) Determine the value, or values, of Q for which $\dfrac{dQ}{dt} = 0$.

Solution: Setting $\dfrac{dQ}{dt} = 0$ in (8) leads to

$$Q = L.$$

$\square$

(b) Find a range of positive values of Q on which $Q(t)$ is increasing, and those values of Q for which $Q(t)$ is decreasing.

Solution: Writing (8) as

$$\frac{dQ}{dt} = \frac{a}{L}\,(L - Q)\,, \tag{9}$$

we see that $\dfrac{dQ}{dt} > 0$ for $Q < L$; so that $Q(t)$ increases for values of Q less than L. Similarly, since $\dfrac{dQ}{dt} < 0$ for $Q > L$, $Q(t)$ decreases for values of Q higher than L. $\square$

(c) Determine values of Q on which the graph of $Q = Q(t)$ is concave up, and those on which it is concave down.

Solution: Differentiating with respect to t on both sided of (9) we obtain

$$\frac{d^2Q}{dt^2} = -\frac{a}{L}\frac{dQ}{dt}, \tag{10}$$

which leads to

$$\frac{d^2Q}{dt^2} = -\frac{a^2}{L^2}(L - Q), \tag{11}$$

by substitution of (9) into (10).

It follows from (11) that $\dfrac{d^2Q}{dt^2} < 0$ for $Q < L$; so that the graph of $Q = Q(t)$ is concave down for values of Q less than L. On the other hand, since $\dfrac{d^2Q}{dt^2} > 0$ for $Q > L$, the graph of $Q = Q(t)$ is concave up for values of Q bigger than L. $\square$

(d) Use the qualitative information obtained in parts (b) and (c) to sketch possible graphs of a solution, $Q = Q(t)$, of the differential equation in (8), for positive values of Q.

Based on your sketches, explain what the equation in (8) seems to be predicting.

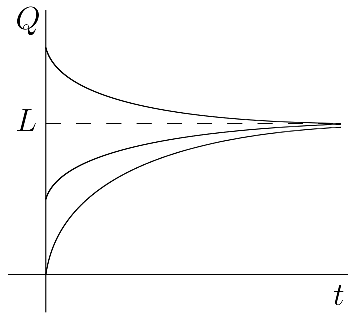

Figure 2: Sketch of graph of $Q(t)$

***Solution**:* Sketches of possible solutions to (8) are shown in Figure 2. The model in (8) seems to be predicting that the quantity $Q(t)$ will tend towards the value $Q = L$. □