Solutions to Assignment #4

1. Solve the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = t\sin(t^2); \\ y(0) = 0, \end{cases}$$

for $t \in \mathbb{R}$.

***Solution**:* Compute

$$y(t) = \int_0^t \tau \sin(\tau^2)\ d\tau,$$

by making the change of variable $u = \tau^2$; so that, $du = 2\tau d\tau$ and

$$\begin{aligned} y(t) &= \frac{1}{2}\int_0^{t^2} \sin(u)\ du \\ &= \frac{1}{2}\left[-\cos(u)\right]_0^{t^2} \\ &= \frac{1}{2}\left[1 - \cos(t^2)\right]. \end{aligned}$$

□

2. Solve the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{\cos(\pi + \sqrt{t})}{\sqrt{t}}; \\ y(\pi^2) = 1, \end{cases}$$

for $t \geqslant 0$.

***Solution**:* Compute

$$y(t) = 1 + \int_{\pi^2}^t \frac{\cos(\pi + \sqrt{\tau})}{\sqrt{\tau}}\ d\tau,$$

by making the change of variable $u = \pi + \sqrt{\tau}$; so that, $du = \dfrac{1}{2\sqrt{\tau}}\, d\tau$ and

$$\begin{aligned} y(t) &= 1 + 2\int_{2\pi}^{\pi+\sqrt{t}} \cos u\ du \\ &= 1 + 2\left[\sin u\right]_{2\pi}^{\pi+\sqrt{t}} \\ &= 1 + 2\sin(\pi + \sqrt{t}). \end{aligned}$$

□

3. Let the graph of $y = f(t)$ be as sketched in Figure 1 on page 2 and put

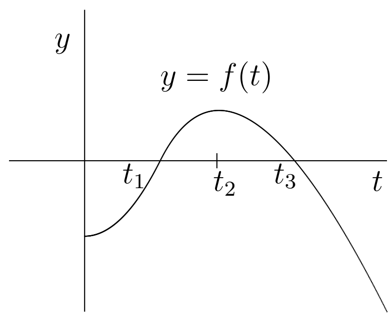

Figure 1: Sketch of graph of $y = f(t)$

$$F(t) = \int_0^t f(\tau)\ d\tau, \text{ for } t \geqslant 0.$$

(a) Based on the sketch in Figure 1, determine intervals on which (i) $F(t)$ increases, (ii) $F(t)$ decreases, (iii) the graph of $y = F(t)$ is concave up, and (iv) the graph of $y = F(t)$ is concave down.

***Solution**:* By the Fundamental Theorem of Calculus, $F' = f$, so that (i) $F(t)$ increases for $t_1 < t < t_3$ and (ii) $F(t)$ decreases for $0 < t < t_1$ and $t > t_3$.
Next, use $F'' = f'$ and the information in the sketch in Figure 1 to conclude that $F'' > 0$ for $0 < t < t_2$ and $F'' < 0$ for $t > t_2$; thus, (iii) the graph of $y = F(t)$ is concave up on the interval $(0, t_2)$ and (iv) concave down for $t > t_2$. □

(b) Estimate the times at which $F(t)$ is (i) a local maximum, and (ii) (i) a local minimum.

Solution: $F(t)$ has a local minimum at $t = t_1$ and a local maximum at $t = t_3$. □

(c) Locate any inflection points in the graph of $y = F(t)$

Solution: The graph of $y = F(t)$ has an inflection point at $(t_2, F(t_2))$. □

4. Let f and F be as in Problem 3. Use the qualitative information obtained in Problem 3 to sketch the graph of $y = F(t)$.

Solution: A sketch of the graph of $y = F(t)$ is shown in Figure 2.

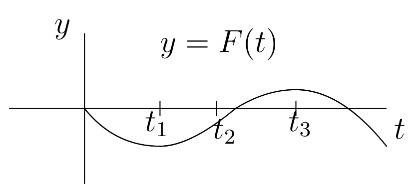

Figure 2: Sketch of graph of $y = F(t)$

□

5. Let f and F be as in Problem 3. Given that $t_2 = 2$ and t_3 is about 3 and $f(t_2)$ is about 0.75, estimate the maximum value of F over the range of values of t pictured in Figure 1.

Solution: In order to do this problem, we also need to estimate $f(0)$ to be about -1; we also estimate t_1 to be about 1.
From the result of part (c) of Problem 3, the maximum of F occurs at $t = t_3$. Thus,

$$\max F = F(t_3).$$

Since $F(t_3) = \displaystyle\int_0^{t_3} f(\tau)\, d\tau,$ we estimate $F(t_3)$ by the negative of the area of the triangle with vertices (0.0), $(0,1)$ and $(t_1, 0)$ plus the area of the triangle with vertices $(t_1, 0)$, $(t_2, 0.75)$ and $(t_3, 0)$. We then have that

$$F(t_3) \approx -\frac{1}{2} + \frac{1}{2} \cdot 2(0.75) = 0.25.$$

□