Assignment #7

Due on Wednesday, October 5, 2016

Read Section 4.3 on *The Number e* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 4.4 on *The Exponential Function* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 1.4 on *Exponential Growth,* pp. 48–55, in *Calculus for the Life Sciences* by Schreiber, Smith and Getz.

Background and Definitions
The exponential function, $\exp\colon \mathbf{R} \to (0, \infty)$, is the unique solution to the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = y; \\ y(0) = 1, \end{cases} \tag{1}$$

for $t \in \mathbf{R}$. We therefore have that

$$\exp'(t) = \exp(t), \qquad \text{for all } t \in \mathbf{R}, \quad \exp(0) = 1,$$

and exp is the only solution to the problem in (1).

Do the following problems

1. Show that $\exp(a - b) = \dfrac{\exp(a)}{\exp(b)}$ for all $a, b \in \mathbf{R}$.

2. Let r and y_o denote real numbers and put $g(t) = y_o \exp(rt)$ for all $t \in \mathbf{R}$. Show that $y = g(t)$ is the unique solution to the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = ry; \\ y(0) = y_o, \end{cases} \tag{2}$$

by considering the function

$$w(t) = \frac{v(t)}{\exp(rt)}, \qquad \text{for all } t \in \mathbf{R},$$

where $v(t)$ is any solution to the initial value problem in (2).

3. Show that
$$\lim_{t\to+\infty} \exp(-t) = 0.$$

4. Define the function $f\colon \mathbf{R} \to \mathbf{R}$ by
$$f(t) = 1 - \exp(-t), \qquad \text{for all } t \in \mathbf{R}.$$
   (a) Compute $f'(t)$ and $f''(t)$.
   (b) Determine the intervals on the t–axis for which f is increasing or decreasing, and all local extrema; the values of t for which the graph of $y = f(t)$ is concave up, and those for which the graph is concave down; and all the inflection points of the graph of $y = f(t)$. Sketch the graph of $y = f(t)$.

5. Let b denote a positive real number. We may use the exponential and natural logarithm functions to define the function $g(t) = b^t$ for all $t \in \mathbf{R}$ as follows
$$g(t) = \exp(t \ln b), \qquad \text{for all } t \in \mathbf{R}. \tag{3}$$
   Use the definition of b^t in (3) to derive formulas for computing
   (i) $\dfrac{d}{dt}[b^t]$, and
   (ii) $\displaystyle\int b^u \, du$.