Review Problems for Exam 1

1. Water leaks out a barrel at a rate proportional to the square root of the depth of the water in the barrel at that time. Derive a differential equation for the depth, $h(t)$, of water in the barrel at time t, given that the cross–sectional area of the barrel is a constant A.

2. A compartment has a fixed volume, V, of isopropyl alcohol solution. A 75% solution of isopropyl alcohol is introduced into the compartment at a rate of $F = 0.1$ liters per minute. Assume that the a well–stirred mixture of the solution flows out of the compartment at the same rate, F.

   (a) Derive a differential equation for the concentration of alcohol, in percent volume, at any time t.

   (b) Sketch possible solutions of the equation.

3. A 1500 gallon tank initially contains 600 gallons of water with 5 lbs of salt dissolved in it. Salt water enters the tank at a rate of 9 gal/hr with a salt concentration of 3 lbs/gal. If a well mixed solution leaves the tank at a rate of 6 gal/hr, derive a differential equation satisfied by the amount of salt in the solution in the tank.

4. Suppose that $y = y(t)$ is a solution of the differential equation

$$\frac{dy}{dt} = -2ty \tag{1}$$

   (a) Assume that $y(t) > 0$ for all t. Determine the values of t for which $y(t)$ increases or decreases.

   (b) Compute y'' in terms of t and y, and determine the values of t for which the graph of $y = y(t)$ is concave up or concave down.

   (c) Given that $y(0) = 1$, use the qualitative information obtained in the previous parts to sketch the graph of $y = y(t)$.

5. Sketch possible solutions of the differential equation

$$\frac{dy}{dt} = (y-1)(y-2).$$

6. In a chemical reaction

$$A + B \to C,$$

let $y(t)$ denote the concentration of the product C at time t. Assume that y is a differentiable function of t. If C_A denote the initial concentration of reactant A and C_B the initial concentration of reactant B, the Law of Mass Action states that

$$\frac{dy}{dt} = k(C_A - y)(C_B - y), \tag{2}$$

where k is a positive constant of proportionality.

Sketch possible solutions of (2) for the case in which $C_A = 40$ and $C_B = 80$.

7. The following equation models the growth of a population that is being harvested at a constant rate:

$$\frac{dN}{dt} = 2N - 0.01N^2 - 75.$$

Sketch possible solutions of the differential equation.

8. Use the Chain Rule to show that $y(t) = y_o \exp(F(t))$, where F is the antiderivative of f with $F(0) = 0$, is a solution of the initial value problem: $\dfrac{dy}{dt} = f(t)y$, $y(0) = y_o$.

9. Evaluate the following integrals

(a) $\displaystyle\int_0^1 \frac{e^{-x}}{2 - e^{-x}}\, \mathrm{d}x$ (b) $\displaystyle\int \frac{1}{x \ln x}\, \mathrm{d}x$

(c) $\displaystyle\int_1^2 \frac{\ln x}{x}\, \mathrm{d}x$ (d) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, \mathrm{d}x$

10. Solve the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{1}{t + \sqrt{t}}; \\ \\ y(1) = 2\ln(2), \end{cases}$$

for $t > 0$.