Solutions to Review Problems for Exam #2

1. Find a solution of the initial value problem $\dfrac{dy}{dt} = e^{t-y}, \quad y(0) = 1.$

   ***Solution**:* Write the differential equation as

   $$\frac{dy}{dt} = e^t e^{-y},$$

   and separate variables to obtain

   $$\int e^y \, dy = \int e^t \, dt,$$

   which integrates to

   $$e^y = e^t + c, \tag{1}$$

   for arbitrary c. Using the initial condition $y(0) = 1$ in (1) yields

   $$e = 1 + c,$$

   from which we get that

   $$c = e - 1. \tag{2}$$

   Substituting the value for c in (2) into the equation in (1) yields

   $$e^y = e^t + e - 1,$$

   which can be solved for y to obtain

   $$y(t) = \ln[e^t + e - 1], \quad \text{for all } t \in \mathbb{R}.$$

   □

2. The temperature in a hot iron decreases at a rate 0.11 times the difference between its present temperature and room temperature (20° C).

   (a) Write a differential equation for the temperature of the iron.

   ***Solution**:* Let $u = u(t)$ denote the temperature of the hot iron at time t. Then,

   $$\frac{du}{dt} = -0.11(u - 20), \tag{3}$$

   where u is measured in degrees Celsius and t in minutes. □

(b) If the initial temperature of the rod is 100° C, and the time is measured in minutes, how long will it take for the rod to reach a temperature of 25°C?

Solution: The general solution of the differential equation in (3) is

$$u(t) = 20 + ce^{-0.11\ t}, \quad \text{for all } t \in \mathbb{R}, \tag{4}$$

for arbitrary constant c.

To find the value of c in (4), we use the initial condition $u(0) = 100$ in (4) to obtain the equation

$$20 + c = 100,$$

which yields

$$c = 80. \tag{5}$$

Substituting the value of c in (5) into the expression for u in (4), we obtain that

$$u(t) = 20 + 80e^{-0.11\ t}, \quad \text{for all } t \in \mathbb{R}. \tag{6}$$

Next, we find the value of t for which $u(t) = 25$, or

$$20 + 80e^{-0.11\ t} = 25,$$

or

$$80e^{-0.11\ t} = 5,$$

which can be solved for t to yield

$$t = -\frac{\ln(1/16)}{0.11} = \frac{4\ln 2}{0.11} \doteq 25 \text{ minutes.}$$

Thus, it will take about 25 minutes for the hot iron to reach the temperature or 25 degrees Celsius. □

3. Consider the first–order ordinary differential equation

$$\frac{dy}{dt} = y^2 - 2y + 1. \tag{7}$$

(a) Determine equilibrium points and determine the nature of the stability of the equilibrium solutions by means of the principle of linearized stability, if applicable.

Solution: Put $f(y) = y^2 - 2y + 1$ and write $f(y) = (y - 1)^2$; so that, the differential equation in (7) has one equilibrium solution; namely,

$$\overline{y} = 1.$$

Since $f'(y) = 2(y - 1)$, $f'(1) = 0$; so that, the principle of linearized stability does not apply in this case. □

(b) Use separation of variables to find the general solution of the equation in (7).

Solution: Use separation of variables to solve the equation

$$\frac{dy}{dt} = (y-1)^2.$$

We obtain

$$\int \frac{1}{(y-1)^2}\, dy = \int dt,$$

which yields

$$-\frac{1}{y-1} = t + c_1, \tag{8}$$

for some arbitrary constant c_1. Multiply on both sides of the equation in (8) by -1 and solve for y to obtain

$$y(t) = 1 + \frac{1}{c-t}, \tag{9}$$

for some arbitrary constant c. □

(c) Use your result from the previous part to determine the nature of the stability of the equilibrium points.

Solution: Let y_o be such that $y_o > 1$, and assume that a solution $y = y(t)$ to the differential equation in (7) satisfies $y(0) = y_o$. We then obtain from (9) that

$$c = \frac{1}{y_o - 1}. \tag{10}$$

Substituting the value for c in (10) into (9) yields the solution

$$y(t) = 1 + \frac{y_o - 1}{1 - (y_o - 1)t} \tag{11}$$

to the initial value problem

$$\begin{cases} \dfrac{dy}{dt} & = & y^2 - 2y + 1; \\ y(0) & = & y_o, \end{cases} \tag{12}$$

which ceases to exist at $t = \dfrac{1}{y_o - 1}$. Therefore, for $y_o > 1$, the solution of the IVP in (12) does not exist for all $t > 0$. Hence, $\overline{y} = 1$ is unstable. □

(d) Find a solution to the IVP $\begin{cases} \dfrac{dy}{dt} &=& y^2 - 2y + 1; \\ y(0) &=& 2, \end{cases}$ and determine its maximal interval of existence.

Solution: Using the formula in (11) derived in the previous part we see that the solution of the IVP in (12) for $y_o = 2$ is given by

$$y(t) = 1 + \frac{1}{1-t}, \quad \text{for } t < 1.$$

Thus, the maximal interval of existence is $(-\infty, 1)$. □

4. Solve the initial value problem $\dfrac{dy}{dt} = y + t^2$, $y(0) = 0$, and compute $\lim\limits_{t\to\infty} y(t)$.

Solution: Rewrite the equation as

$$\frac{dy}{dt} - y = t^2$$

and multiply by the integrating factor e^{-t} to obtain

$$e^{-t}\frac{dy}{dt} - e^{-t}y = t^2e^{-t},$$

which can be written as

$$\frac{d}{dt}[e^{-t}y] = t^2e^{-t}, \tag{13}$$

by virtue of the product rule. Integrating on both sides of (13) yields

$$e^{-t}y = \int t^2e^{-t}\ dt. \tag{14}$$

In order to evaluate the integral on the right–hand side of (14), we use integration by parts.

Let

$$u = t^2 \qquad \text{and} \qquad dv = e^{-t}\ dt;$$

so that,

$$du = 2t\ dt \qquad \text{and} \qquad v = -e^{-t}.$$

Then,

$$\int t^2e^{-t}\ dt = -t^2e^{-t} + \int 2te^{-t}\ dt. \tag{15}$$

The right–most integral in (15) can also be evaluated using integration by parts.

$$u = 2t \qquad \text{and} \qquad dv = e^{-t}\, dt;$$

so that

$$du = 2\, dt \qquad \text{and} \qquad v = -e^{-t},$$

and, therefore,

$$\int 2te^{-t}\, dt = -2te^{-t} + \int 2e^{-t}\, dt,$$

from which we get that

$$\int 2te^{-t}\, dt = -2te^{-t} - 2e^{-t} + c, \tag{16}$$

for some constant of integration c. Substituting the result in (16) into (15) then yields

$$\int t^2 e^{-t}\, dt = -t^2 e^{-t} - 2te^{-t} - 2e^{-t} + c, \tag{17}$$

where c is an arbitrary constant. Substituting the result in (17) into the right–hand side of (14) yields

$$e^{-t} y = -(t^2 + 2t + 2)e^{-t} + c \tag{18}$$

Solving for y in (18) we obtain

$$y(t) = -t^2 - 2t - 2 + ce^t, \qquad \text{for all } t \in \mathbb{R}. \tag{19}$$

Using the initial condition, $y(0) = 0$, in (18) we obtain that $-2 + c = 0$, we have that $c = 2$. Thus,

$$y(t) = 2e^t - t^2 - 2t - 2, \qquad \text{for all } t \in \mathbb{R}. \tag{20}$$

It follows from (20) that $\lim_{t\to\infty} y(t) = +\infty$. □

5. **Logistic Growth with Harvesting.** The following differential equation models the growth of a population of size $N = N(t)$ that is being harvested at a rate proportional to the population density

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - EN, \tag{21}$$

where r, K and E are non–negative parameters with $r > 0$ and $K > 0$.

(a) Give an interpretation for this model. In particular, give interpretation for the term EN. The parameter E is usually called the harvesting *effort.*

> ***Answer:*** This equation models a population that grows logistically and that is also being harvested at a rate proportional to the populations density. □

(b) Calculate the equilibrium points for the equation (21), and give conditions on the parameters that yield a biologically meaningful equilibrium point. Determine the nature of the stability of that equilibrium point. Sketch possible solutions to the equation in this situation.

Solution: Write

$$\begin{aligned} f(N) &= rN\left(1 - \frac{N}{K}\right) - EN \\ &= rN\left(1 - \frac{N}{K} - \frac{E}{r}\right) \\ &= -\frac{r}{K}N\left[N - K\left(1 - \frac{E}{r}\right)\right]. \end{aligned}$$

We then see that equilibrium points of equation (21) are

$$\overline{N}_1 = 0 \quad \text{and} \quad \overline{N}_2 = K\left(1 - \frac{E}{r}\right). \tag{22}$$

The second equilibrium point is biologically meaningful if $\overline{N}_2 > 0$, and for this to happen we require that $E < r$; that is, the harvesting effort is less than the intrinsic growth rate.

To determine the nature of the stability of $\overline{N}_2$ for the case $E < r$, consider a sketch of the graph of $f(N)$ versus N in Figure 1. Observe from the sketch that $f'(\overline{N}_2) < 0$. It then follows from the principle of linearized stability that $\overline{N}_2$ is asymptotically stable.

The solid curves in Figure 2 show possible solutions of the equation. □

(c) What does the model predict if $E \geqslant r$?

Solution: If $E = r$, then

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\frac{r}{K}N^2 < 0$$

for $N > 0$. It then follows that $N(t)$ will always be strictly decreasing and so the population will go extinct. In fact, using separation of variables, we

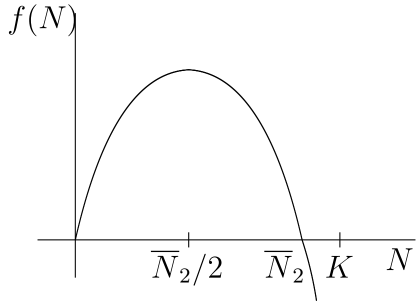

Figure 1: Graph of $f(N)$ versus N

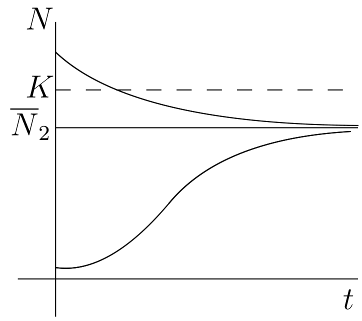

Figure 2: Possible Solutions

obtain that the solution for $N(0) = N_o$ is given by

$$N(t) = \frac{N_o K}{K + N_o r t},$$

which tends to 0 as $t \to \infty$.
On the other hand, if $E > r$, then

$$\begin{aligned} \frac{\mathrm{d}N}{\mathrm{d}t} &= -\frac{r}{K} N \left[N - K\left(1 - \frac{E}{r}\right)\right] \\ &= -\frac{r}{K} N^2 + KN(r - E) \\ &< -\frac{r}{K} N^2 < 0, \end{aligned}$$

and so again we conclude the $N(t)$ will be always decreasing to 0. □