Assignment #17

Due on Monday, November 27, 2017

Read Section 8.1 on the *Definition of Convergence in Distribution* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 8.2 on the *mgf Convergence Theorem* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 8.3 on the *Central Limit Theorem* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Read Section 5.4 on *The Poisson Distribution* in DeGroot and Schervish.

Read Section 5.6 on *The Normal Distribution* in DeGroot and Schervish.

Read Section 6.3 on *The Central Limit Theorem* in DeGroot and Schervish.

Background and Definitions

Definition (Convergence in Distribution)**.** Let (X_n) be a sequence of random variables with cumulative distribution functions F_{X_n}, for $n = 1, 2, 3, \ldots$, and Y be a random variable with cdf F_Y. We say that the sequence (X_n) converges to Y in distribution, if

$$\lim_{n\to\infty} F_{X_n}(x) = F_Y(x)$$

for all x where F_Y is continuous. The distribution of Y is usually called the **limiting distribution** of the sequence (X_n).

Theorem (mgf Convergence Theorem)**.** *Let (X_n) be a sequence of random variables with moment generating functions $\psi_{X_n}(t)$, for $|t| < h$, $n = 1, 2, 3, \ldots$, and some positive number h. Suppose Y has mgf $\psi_Y(t)$ which exists for $|t| < h$. Then, if*

$$\lim_{n\to\infty} \psi_{X_n}(t) = \psi_Y(t), \quad \text{for } |t| < h,$$

it follows that $\lim\limits_{n\to\infty} F_{X_n}(x) = F_Y(x)$ for all x where F_Y is continuous.

Do the following problems

1. An experiment consists of rolling a die 81 times and computing the average of the numbers on the top face of the die. Estimate the probability that the sample mean will be less than 3.

2. Let (X_k) denote a sequence of independent identically distributed random variables such that $X_k \sim \text{Normal}(\mu, \sigma^2)$ for every $k = 1, 2, \ldots$, and for some $\mu \in \mathbb{R}$ and $\sigma > 0$.

   (a) For each $n \geqslant 1$, define $\overline{X}_n = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$.

   Determine the mgf, $\psi_{\overline{X}_n}(t)$, for $\overline{X}_n$, and compute $\lim\limits_{n\to\infty} \psi_{\overline{X}_n}(t)$.

   Give the limiting distribution of $\overline{X}_n$ as $n \to \infty$.

   (b) Define $Z_n = \dfrac{\overline{X}_n - \mu}{\sigma/\sqrt{n}}$ for all $n \geqslant 1$.

   Determine the mgf, $\psi_{Z_n}(t)$, for Z_n, and compute $\lim\limits_{n\to\infty} \psi_{Z_n}(t)$.

   Find the limiting distribution of Z_n as $n \to \infty$.

3. Let $Y_n \sim \text{Binomial}(n, p)$, for $n = 1, 2, 3, \ldots$, and define $Z_n = \dfrac{Y_n - np}{\sqrt{np(1-p)}}$ for $n = 1, 2, 3, \ldots$ Use the Central Limit Theorem to find the limiting distribution of Z_n.

   Suggestion: Recall that Y_n is the sum of n independent Bernoulli(p) trials.

4. Suppose that a random sample of size n is to be taken from a distribution for which the mean is μ and the standard deviation is 3. Use the Central Limit Theorem to determine approximately the smallest value of n for which the following relation will be satisfied: $\Pr(|\overline{X}_n - \mu| < 0.3) \geqslant 0.95$.

5. Forty–nine measurements are recorded to several decimal places. Each of these 49 numbers is rounded off to the nearest integer. The sum of the original 49 numbers is approximated by the sum of those integers. Assume that the errors made in rounding off are independent, identically distributed random variables with a uniform distribution over the interval $(-0.5, 0.5)$. Compute approximately the probability that the sum of the integers is within two units of the true sum.