## Assignment #3

**Due on Friday, September 22, 2017**

**Read** Section 3.2, *Fundamental Lemmas,* in the class lecture notes at
http://pages.pomona.edu/~ajr04747/

**Read** Section 3.3, *The Euler–Lagrange Equation,* in the class lecture notes at
http://pages.pomona.edu/~ajr04747/

**Read** Sections 2–12, 2–13, 2–14, 3–1, 3–2 and 3–3, pp. 12–24, in *Calculus of Variations* by Robert Weinstock.

**Do** the following problems

1. Let $\mathcal{A} = \{y \in C^1([0,1],\mathbb{R}) \mid y(0) = 0 \text{ and } y(1) = 1\}$. Define $J\colon \mathcal{A} \to \mathbb{R}$ by

$$J(y) = \int_0^1 [2e^x y(x) + (y'(x))^2]\, \mathrm{d}x \quad \text{for all } y \in \mathcal{A}.$$

   Give the Euler–Lagrange equation associated with $J$ and, if possible, solve it subject to the boundary conditions in $\mathcal{A}$.

2. Let $J\colon \mathcal{A} \to \mathbb{R}$ be defined by $J(y) = \displaystyle\int_1^2 [2(y(x))^2 + x^2(y'(x))^2]\, \mathrm{d}x$ for all $y \in \mathcal{A}$, where $\mathcal{A} = \{y \in C^1([1,2],\mathbb{R}) \mid y(1) = 1 \text{ and } y(2) = 5\}$. Give the Euler–Lagrange equation associated with $J$ and, if possible, solve it subject to the boundary conditions in $\mathcal{A}$.

3. Let $J\colon \mathcal{A} \to \mathbb{R}$ be defined by $J(y) = \displaystyle\int_5^{10} \sqrt{x}\sqrt{1 + (y'(x))^2}\, \mathrm{d}x$ for all $y \in \mathcal{A}$, where $\mathcal{A} = \{y \in C^1[5,10] \mid y(5) = 4 \text{ and } y(10) = 16\}$. Give the Euler–Lagrange equation associated with $J$ and, if possible, solve it subject to the boundary conditions in $\mathcal{A}$.

4. **The Brachistochrone Problem**. The Euler–Lagrange equation associated with the functional defined in the discussion of the Brachistochrone problem in class, and in the lecture notes, can be written in the form

$$\frac{d}{dx}\left[\frac{u'}{\sqrt{1 + (u')^2}\sqrt{u}}\right] = -\frac{\sqrt{1 + (u')^2}}{2u^{3/2}}, \quad \text{for } 0 < x < x_1, \tag{1}$$

Evaluate the derivative on the left–hand side of the equation in (1) and simplify to obtain from (1) that

$$(u')^2 + 2uu'' + 1 = 0 \quad \text{for } 0 < x < x_1, \tag{2}$$

where $u''$ denotes the second derivative of $u$.

5. **The Brachistochrone Problem, Continued**. Multiply on both sides of (2) by $u'$ to get

$$(u')^3 + 2uu'u'' + u' = 0 \quad \text{for } 0 < x < x_1. \tag{3}$$

   (a) Show that the differential equation in (3) can be written as

$$\frac{d}{dx}[u + u(u')^2] = 0. \tag{4}$$

   (b) Integrate the equation in (4) to obtain a first–order differential equation for $u$.