Assignment #5

Due on Friday, October 6, 2017

Read Section 4.2, *A Minimization Problem,* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Read Section 4.3, *Convex Functionals,* in the class lecture notes at
`http://pages.pomona.edu/~ajr04747/`

Background and Definitions

Convex Functionals.
Let V denote a normed linear space, V_o a nontrivial subspace of V, and $\mathcal{A}$ a given nonempty subset of V. Let $J\colon V \to \mathbb{R}$ be a functional defined on V. Suppose that J is Gâteaux differentiable at every $u \in \mathcal{A}$ in any direction $v \in V_o$. The functional J is said to be **convex** on $\mathcal{A}$ if

$$J(u+v) \geqslant J(u) + dJ(u;v)$$

for all $u \in \mathcal{A}$ and $v \in V_o$ such that $u + v \in \mathcal{A}$.

A Gâteaux differentiable functional $J\colon V \to \mathbb{R}$ is said to be **strictly convex** in $\mathcal{A}$ if it is convex in $\mathcal{A}$, and

$$J(u+v) = J(u) + dJ(u;v), \quad \text{for } u \in \mathcal{A}, v \in V_o \text{ with } u+v \in \mathcal{A}, \text{ iff } v = 0.$$

Do the following problems

1. Let Ω denote an open subset of $\mathbb{R}^n$ and $u\colon \overline{\Omega} \to \mathbb{R}$ a continuous function. Suppose also that $u(x) \geqslant 0$ for all $x \in \Omega$ and that

   $$\int_\Omega u(x)\, dx = 0.$$

   Show that $u(x) = 0$ for all $x \in \overline{\Omega}$

2. Let U denote an open subset of $\mathbb{R}^n$. We say that U is **path connected** if and only if for any two points x_o and x_1 in U, there exists a differentiable path $\sigma\colon [a,b] \to U$ such that

   $$\sigma(0) = x_o \quad \text{and} \quad \sigma(1) = x_1.$$

   Let $v \in C^1(U, \mathbb{R})$, where U is path connected. Suppose that

   $$\nabla v(x) = 0, \quad \text{for all } x \in U.$$

   Show that v must be constant in U.

3. Use the Cauchy–Schwarz inequality in $\mathbb{R}^2$ applied to the vectors $\vec{A} = (1, z)$ and $\vec{B} = (1, z + w)$ to deduce the inequality

$$\sqrt{1 + (z + w)^2} \geqslant \sqrt{1 + z^2} + \frac{zw}{\sqrt{1 + z^2}},$$

with equality if and only if $w = 0$.

Use this fact to show that the arc–length functional,

$$J(y) = \int_a^b \sqrt{1 + (y'(x))^2}\ dx, \qquad \text{for all } y \in C^1([a, b], \mathbb{R}),$$

is strictly convex.

4. Let $V = C([a, b], \mathbb{R})$ and define $J : V \to \mathbb{R}$ by

$$J(y) = \int_a^b (\sin^3 x + y^2(x))\ \mathrm{d}x \quad \text{for all } y \in V.$$

   (a) Show that J is Gateaux differentiable and compute $dJ(y; v)$ for all $y, v \in V$.

   (b) Show that J is strictly convex.

5. Let V be a normed linear space and $L : V \to \mathbb{R}$ be a linear functional.

   Show that J is convex but not strictly convex.