

Assignment #6

Due on Friday, October 27, 2017

Read Chapter 5, *Optimization Problems with Constraints*, in the class lecture notes at <http://pages.pomona.edu/~ajr04747/>

Read Chapter 4, *Isoperimetric Problems*, in *Calculus of Variations* by Robert Weinstock.

Background and Definitions

Weak Continuity.

Let V denote a normed linear space with norm $\|\cdot\|$ and V_o a nontrivial subspace of V . Let $J: V \rightarrow \mathbb{R}$ be Gâteaux differentiable at every $u \in V$ in the direction of every $v \in V_o$. We say that the Gâteaux derivative of J , $dJ(u; v)$, is weakly continuous at $u_o \in V$ if and only if

$$\lim_{u \rightarrow u_o} dJ(u; v) = dJ(u_o; v), \quad \text{for every } v \in V_o;$$

that is, for each $v \in V_o$, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|u - u_o\| < \delta \Rightarrow |dJ(u; v) - dJ(u_o; v)| < \varepsilon.$$

Euler–Lagrange Multiplier Theorem.

Let V denote a normed linear space and V_o a nontrivial subspace of V . Let $J: V \rightarrow \mathbb{R}$ and $K: V \rightarrow \mathbb{R}$ be functionals that are Gâteaux differentiable at every $u \in V$ in the direction of every $v \in V_o$. Suppose there exists $u_o \in V$ such that

$$K(u_o) = c,$$

for some real number c , and

$$J(u_o) \leq J(v) \text{ (or } J(u_o) \geq J(v)) \text{ for all } v \in V \text{ such that } K(v) = c.$$

Suppose also that the Gâteaux derivatives, $dJ(u; v)$ and $dK(u; v)$, of J and K , respectively, are weakly continuous in u for all $u \in V$. Then, either

$$dK(u_o; v) = 0, \quad \text{for all } v \in V_o,$$

or there exists a real number μ such that

$$dJ(u_o; v) = \mu dK(u_o; v), \quad \text{for all } v \in V_o.$$

Do the following problems

1. Let $V = C^1([a, b], \mathbb{R})$ and define

$$\|y\| = \max_{a \leq x \leq b} |y(x)| + \max_{a \leq x \leq b} |y'(x)|, \quad \text{for all } y \in V. \quad (1)$$

Verify that $\|\cdot\|$ given in (1) defines a norm in V .

2. Let $V = C^1([a, b], \mathbb{R})$. We consider V with the norm given in (1). Define $J: V \rightarrow \mathbb{R}$ by

$$J(y) = \int_a^b \sqrt{1 + (y'(x))^2} dx, \quad \text{for all } y \in V. \quad (2)$$

Verify that the Gâteaux derivative of the functional J defined in (2) is weakly continuous at all $y \in V$.

Suggestion: Define $g(z) = \frac{z}{\sqrt{1+z^2}}$, for all $z \in \mathbb{R}$, and show that

$$|g(z_1) - g(z_2)| \leq |z_1 - z_2|, \quad \text{for all } z_1, z_2 \in \mathbb{R}.$$

Hint: Verify that $|g'(z)| \leq 1$ for all $z \in \mathbb{R}$.

3. Let V be a normed linear space with norm $\|\cdot\|$. A linear functional $L: V \rightarrow \mathbb{R}$ is said to be bounded if and only if there exists a constant $M > 0$ such that

$$|L(v)| \leq M\|v\|, \quad \text{for all } v \in V.$$

Let $J: V \rightarrow \mathbb{R}$ be functional defined by $J(u) = L(u)$ for all $u \in V$, where $L: V \rightarrow \mathbb{R}$ is bounded linear functional. Show that the Gâteaux derivative of J is weakly continuous for all $u \in V$.

4. Consider the class \mathcal{A} of functions $y \in C_o^1([0, b], \mathbb{R})$ such that $y(x) \geq 0$ for all $x \in [0, b]$ and

$$\int_0^b y(x) dx = a,$$

for a positive value a .

Give the necessary condition that $y \in \mathcal{A}$ must satisfy in order for functional $J: V \rightarrow \mathbb{R}$ given by

$$J(y) = \int_0^b \sqrt{1 + (y'(x))^2} dx, \quad \text{for all } y \in C_o^1([0, b], \mathbb{R}),$$

to be the smallest possible in \mathcal{A} at y .

Explain the reasoning leading to your answer.

5. Let $V = C^1([a, b], \mathbb{R})$ be endowed with the norm $\|\cdot\|$ defined in (1). Let $F: [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $G: [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions with continuous partial derivatives F_y, F_z, G_y and G_z .

Define the functionals $J: V \rightarrow \mathbb{R}$ and $K: V \rightarrow \mathbb{R}$ by

$$J(y) = \int_a^b F(x, y(x), y'(x)) \, dx, \quad \text{for all } y \in V,$$

and

$$K(y) = \int_a^b G(x, y(x), y'(x)) \, dx, \quad \text{for all } y \in V.$$

Assume that the Gâteaux derivatives of J and K are weakly continuous in y .

Apply the Euler–Lagrange Multiplier Theorem to find necessary conditions for $y \in V$ to be an optimizer of the functional J over the constraint $K(y) = c$, for some real constant c .