

Assignment #17

Due on Wednesday, November 28, 2018

Read Section 11.2 on *Differential 1-Forms*, pp. 523–526, in Baxandall and Liebek's text.

Read Section 5.5 on *Differential Forms* in the class Lecture Notes at <http://pages.pomona.edu/~ajr04747/>.

Background and Definitions

- (*Differential 0-Forms*) A differential 0 form in an open set $U \subseteq \mathbb{R}^n$ is a C^∞ function, $f: U \rightarrow \mathbb{R}$. A differential 0-form acts on points, p , in U by means of function evaluation: $f_p = f(p)$, for all $p \in U$.
- (*Differential 1-Forms*) Let U denote an open subset of \mathbb{R}^n and let $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$ denote the space of real valued linear transformations defined in \mathbb{R}^n . A differential 1-form, ω , on U is a (smooth) map $\omega: U \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R})$ which assigns to each $p \in U$, a linear transformation $\omega_p: \mathbb{R}^n \rightarrow \mathbb{R}$ given by

$$\omega_p(h) = F_1(p)h_1 + F_2(p)h_2 + \cdots + F_n(p)h_n,$$

for all $h = (h_1, h_2, \dots, h_n) \in \mathbb{R}^n$, where the vector field $F = (F_1, F_2, \dots, F_n)$ is a smooth vector field.

Differential 1 forms act on oriented, smooth curves, C , by means on integration; we write

$$\omega(C) = \int_C \omega = \int_C F_1 dx_1 + F_2 dx_2 + \cdots + F_n dx_n.$$

Do the following problems

1. Evaluate the differential 1-form $\omega = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$ in the directed line segment from $P_0(1, 1)$ to $P_1(0, 1)$.
2. A differential 1-form, ω , is said to be exact if there exists a 0-form, f , such that $\omega = df$ in the domain of definition of f and ω . Determine which of the following 1-forms are exact.
 - (a) $yz dx + xz dy + xy dz$

(b) $xy \, dx + yz \, dy + xz \, dz$

(c) $(2xyz + z) \, dx + (x^2z + 1) \, dy + (x^2y + x) \, dz$

3. Show that a differential 1-form

$$\omega = F_1 \, dx_1 + F_2 \, dx_2 + \cdots + F_n \, dx_n$$

is exact if and only if the vector field $F = F_1 \, e_1 + F_2 \, e_2 + \cdots + F_n \, e_n$ is the gradient of a smooth function $f: U \rightarrow \mathbb{R}$.

4. Let $\omega = -y \, dx + x \, dy$. Evaluate the differential 1-form on the unit circle, C , oriented in the counterclockwise sense.

5. Let ω denote a differential 1-form in \mathbb{R}^n , and let P_1 and P_2 be any two points in \mathbb{R}^n . Show that

$$\int_{[P_1, P_2]} \omega = - \int_{[P_2, P_1]} \omega.$$