Assignment #18

Due on Monday, December 3, 2018

Read Section 11.3 on *Differential* 2*–Forms,* pp. 527–534, in Baxandall and Liebek's text.

Read Section 5.5 on *Differential Forms* in the class Lecture Notes at `http://pages.pomona.edu/~ajr04747/`.

Background and Definitions

- *(Skew–Symmetric, Bilinear Forms in* $\mathbb{R}^n$*)* A skew–symmetric bilinear form in $\mathbb{R}^n$ is a map, $B\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, which assigns to each pair of vectors, v and w, in $\mathbb{R}^n$, a real value, $B(v,w)$; the form $B(v,w)$ is linear in both v and w; and
$$B(w,v) = -B(v,w), \quad \text{for all } v,w \in \mathbb{R}^n.$$
Denote by $\mathcal{A}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ the space of skew–symmetric, bilinear forms.

- *(Differential* 2*–Forms in* $\mathbb{R}^n$*)* A differential 2–form in an open set $U \subseteq \mathbb{R}^n$ is a smooth function, $\omega\colon U \to \mathcal{A}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$, which assigns to each $p \in U$ a skew–symmetric, bilinear form $\omega_p\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

- *(Wedge Product of Differential* 1*–Forms)* Given two differential 1–forms, ω and η, in some open subset, U, of $\mathbb{R}^n$, the wedge product of ω and η is the differential 2–form in U, denoted by $\omega \wedge \eta$ and defined by
$$(\omega \wedge \eta)_p(v,w) = \omega_p(v)\eta_p(w) - \omega_p(w)\eta_p(v), \quad \text{for } p \in U, \text{ and } v,w \in \mathbb{R}^n.$$

Do the following problems

1. Given $v = a_1\,\widehat{i} + a_2\,\widehat{j}$ and $w = b_1\,\widehat{i} + b_2\,\widehat{j}$ in $\mathbb{R}^2$, compute the wedge product of dx and dy at (v,w); that is, evaluate $dx \wedge dy(v,w)$. What do you conclude?

2. Let U denote an open subset of $\mathbb{R}^2$. Prove that any differential 2–form, ω, in U must be of the form
$$\omega = f(x,y)\; dx \wedge dy, \quad \text{for all } (x,y) \in U, \tag{1}$$
where $f\colon U \to \mathbb{R}$ is a smooth scalar field on U.

   Suggestion: Begin with an arbitrary differential 2–form, ω, in U, and evaluate $\omega_p(v,w)$ for arbitrary points $p \in U$ and arbitrary pairs of vectors $v = a_1\,\widehat{i} + a_2\,\widehat{j}$ and $w = b_1\,\widehat{i} + b_2\,\widehat{j}$ in $\mathbb{R}^2$.

3. Express the following wedge products of differential 1–forms in $\mathbb{R}^2$ in the standard form given in (1). In each case, identify the function f in (1).

   (a) $(dx + dy) \wedge (dx - dy)$;

   (b) $(x\ dx + y\ dy) \wedge (y\ dx + x\ dy)$;

4. Let $A = [a_{ij}]$ denote a 2×2 matrix, and define the differential 1–forms in $\mathbb{R}^2$:

$$\omega_1 = a_{11}\ dx + a_{21}\ dy$$

   and

$$\omega_2 = a_{12}\ dx + a_{22}\ dy.$$

   Compute $\omega_1 \wedge \omega_2$. What do you conclude?

5. Express the differential 2–form in $\mathbb{R}^3$

$$x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy$$

   as a wedge product of differential 1–forms in $\mathbb{R}^3$.