Assignment #6

Due on Friday, October 25, 2019

Read Chapter 5, *Optimization Problems with Constraints,* in the class lecture notes at `http://pages.pomona.edu/~ajr04747/`

Background and Definitions

Weak Continuity.
Let V denote a normed linear space with norm $\|\cdot\|$ and V_o a nontrivial subspace of V. Let $J\colon V\to\mathbb{R}$ be Gâteaux differentiable at every $u\in V$ in the direction of every $v\in V_o$. We say that the Gâteaux derivative of J, $dJ(u;v)$, is weakly continuous at $u_o\in V$ if and only if

$$\lim_{u\to u_o} dJ(u;v) = dJ(u_o;v), \quad \text{for every } v\in V_o.$$

This is equivalent to

$$\lim_{\|u-u_o\|\to 0} |dJ(u;v) - dJ(u_o;v)| = 0, \quad \text{for every } v\in V_o.$$

Euler–Lagrange Multiplier Theorem.
Let V denote a normed linear space and V_o a nontrivial subspace of V. Let $J\colon V\to\mathbb{R}$ and $K\colon V\to\mathbb{R}$ be functionals that are Gâteaux differentiable at every $u\in V$ in the direction of every $v\in V_o$. Assume also that the Gâteaux derivatives, $dJ(u;v)$ and $dK(u;v)$, of J and K, respectively, are weakly continuous in u for all $u\in V$.

Suppose there exists $u_o\in V$ such that $K(u_o)=c$, for some real number c, and

$$J(u_o)\leqslant J(v) \text{ (or } J(u_o)\geqslant J(v)) \text{ for all } v\in V \text{ such that } K(v)=c.$$

Then, either

$$dK(u_o;v)=0, \quad \text{for all } v\in V_o,$$

or there exists a real number μ such that

$$dJ(u_o;v)=\mu\, dK(u_o;v), \quad \text{for all } v\in V_o.$$

Do the following problems

1. Let $V=C^1([a,b],\mathbb{R})$ and define

$$\|y\| = \max_{a\leqslant x\leqslant b}|y(x)| + \max_{a\leqslant x\leqslant b}|y'(x)|, \quad \text{for all } y\in V. \tag{1}$$

   Verify that $\|\cdot\|$ given in (1) defines a norm in V.

2. Let $V = C^1([a,b],\mathbb{R})$. We consider V with the norm given in (1).

   Define $J\colon V \to \mathbb{R}$ by

$$J(y) = \int_a^b \sqrt{1 + (y'(x))^2}\, dx, \quad \text{for all } y \in V. \tag{2}$$

   Verify that the Gâteaux derivative of the functional J defined in (2) is weakly continuous at all $y \in V$.

   Suggestion: Define $g(z) = \dfrac{z}{\sqrt{1+z^2}}$, for all $z \in \mathbb{R}$, and show that

$$|g(z_1) - g(z_2)| \leqslant |z_1 - z_2|, \quad \text{for all } z_1, z_2 \in \mathbb{R}.$$

   Hint: Verify that $|g'(z)| \leqslant 1$ for all $z \in \mathbb{R}$.

3. Let V be a normed linear space with norm $\|\cdot\|$. A linear functional $L\colon V \to \mathbb{R}$ is said to be bounded if and only if there exists a constant $M > 0$ such that

$$|L(v)| \leqslant M\|v\|, \quad \text{for all } v \in V.$$

   Let $J\colon V \to \mathbb{R}$ be functional defined by $J(u) = L(u)$ for all $u \in V$, where $L\colon V \to \mathbb{R}$ is a bounded linear functional. Show that the Gâteaux derivative of J is weakly continuous for all $u \in V$.

4. Consider the class $\mathcal{A}$ of functions $y \in C_o^1([0,b],\mathbb{R})$ such that $y(x) \geqslant 0$ for all $x \in [0,b]$ and $\displaystyle\int_0^b y(x)\, dx = a$, for a positive value a.

   Give the necessary condition that $y \in \mathcal{A}$ must satisfy for functional $J\colon V \to \mathbb{R}$ given by

$$J(y) = \int_0^b \sqrt{1 + (y'(x))^2}\, dx, \quad \text{for all } y \in C_o^1([0,b],\mathbb{R}),.$$

   to be the smallest possible in $\mathcal{A}$ at y. Explain the reasoning leading to your answer.

5. Let $V = C^1([a,b],\mathbb{R})$ be endowed with the norm $\|\cdot\|$ defined in (1). Let $F\colon [a,b]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $G\colon [a,b]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be continuous functions with continuous partial derivatives F_y, F_z, G_y and G_z.

Define the functionals $J\colon V \to \mathbb{R}$ and $K\colon V \to \mathbb{R}$ by

$$J(y) = \int_a^b F(x, y(x), y'(x))\, dx, \quad \text{for all } y \in V,$$

and

$$K(y) = \int_a^b G(x, y(x), y'(x))\, dx, \quad \text{for all } y \in V.$$

Assume that the Gâteaux derivatives of J and K are weakly continuous in y.

Apply the Euler–Lagrange multiplier theorem to find necessary conditions for $y \in V$ to be an optimizer of the functional J subject to the constraint $K(y) = c$, for some real constant c.