Assignment #15

Due on Wednesday, December 4, 2019

Read Section 5.2 on *Line Integrals* in the class Lecture Notes at
`http://pages.pomona.edu/~ajr04747/`.

Background and Definitions

- *(Line Integrals)* Let U be an open subset of $\mathbb{R}^n$ and $F\colon U \to \mathbb{R}^n$ be a continuous vector field. Let $C \subset U$ be a C^1 simple curve parmetrized by a C^1 path $\sigma\colon [a,b] \to \mathbb{R}^n$. We define the line integral of F over C, oriented according to the parametrization, σ, denoted $\displaystyle\int_C F \cdot d\overrightarrow{r}$, to be

$$\int_C F \cdot d\overrightarrow{r} = \int_a^b F(\sigma(t)) \cdot \sigma'(t)\ dt.$$

If $U \subseteq \mathbb{R}^3$ and $F = f_1\widehat{i} + f_2\widehat{j} + f_3\widehat{k}$, where $f_j\colon U \to \mathbb{R}$ are continuous scalar fields, we denote $\displaystyle\int_C F \cdot d\overrightarrow{r}$ by $\displaystyle\int_C f_1\ dx + f_2\ dy + f_3\ dz$. The expression $f_1\ dx + f_2\ dy + f_3\ dz$ is called a differential 1–form in $\mathbb{R}^3$.

- If the curve C is not C^1, but is piece–wise C^1, then the line integral of F over C is given by:

$$\int_C F \cdot d\overrightarrow{r} = \sum_{i=i}^{k} \int_{C_i} F \cdot d\overrightarrow{r},$$

where $C = \displaystyle\bigcup_{i=1}^{k} C_i$, and the orientation of each C_i is consistent with that of C.

Do the following problems

1. Consider a portion of a helix, C, parametrized by the path

$$\sigma(t) = (\cos t, t, \sin t) \quad \text{for} \quad 0 \leqslant t \leqslant \pi.$$

Let $F(x,y,z) = x\ \widehat{i} + y\ \widehat{j} + z\ \widehat{k}$, for all $(x,y,z) \in \mathbb{R}^3$, be a vector field in $\mathbb{R}^3$. Evaluate the line integral $\displaystyle\int_C F \cdot d\overrightarrow{r}$; that is, the integral of the tangential component of the field F along the curve C.

2. Evaluate $\int_C yz\ dx + xz\ dy + xy\ dz$, where C is the directed line segment from the point $(1, 1, 0)$ to the point $(3, 2, 1)$ in $\mathbb{R}^3$.

3. Integrate the 1–form $xy^2\ dx + y\ dy$ along each of the following paths from $(0, 0)$ to $(1, 1)$:

   (a) the straight line form $(0, 0)$ to $(1, 1)$,

   (b) the line from $(0, 0)$ to $(1, 0)$ followed by the line from $(1, 0)$ to $(1, 1)$,

   (c) the lines from $(0, 0)$ to $(0, 1)$ to $(1, 1)$.

4. Integrate the 1–form $xy^2\ dx + y\ dy$ along each of the following paths from $(0, 0)$ to $(1, 1)$:

   (a) the curve $y = x^2$;

   (b) the curve $x = y^2$;

   (c) the lines from $(0, 0)$ to $(2, 0)$ to $(2, 1)$ to $(1, 1)$.

5. Let $f: U \to \mathbb{R}$ be a C^1 scalar field defined on an open subset U of $\mathbb{R}^n$. Define the vector field $F: U \to \mathbb{R}^n$ by $F(u) = \nabla f(u)$ for all $u \in U$. Suppose that C is a C^1 simple curve in U connecting the point u to the point v in U. Show that

$$\int_C F \cdot d\overrightarrow{r} = f(v) - f(u).$$

   Conclude therefore that the line integral of F along a path from u to v in U is independent of the path connecting u to v. The field F is called a *gradient field.*