Assignment #16

Due on Friday, December 6, 2019

Read Section 5.3 on *Gradient Fields* in the class Lecture Notes at `http://pages.pomona.edu/~ajr04747/`.

Read Section 5.4 on *Flux Across Plane Curves* in the class Lecture Notes at `http://pages.pomona.edu/~ajr04747/`.

Read Section 5.7 on *Evaluating Differential 2–Forms: Double Integrals* in the class Lecture Notes at `http://pages.pomona.edu/~ajr04747/`.

Background and Definitions

- **Flux across a simple, closed curve in $\mathbb{R}^2$.** Let U denote an open subset of $\mathbb{R}^2$ and $F\colon U \to \mathbb{R}^2$ be a two–dimensional vector field given by

$$F(x,y) = P(x,y)\,\widehat{i} + Q(x,y)\,\widehat{j}, \quad \text{for all } (x,y)\in U,$$

where P and Q are scalar fields defined in U. Let C denote a simple, piece–wise C^1, closed curve contained in U, which is oriented in the counterclockwise sense. The flux of F across C, denoted by $\oint_C F\cdot\widehat{n}\; ds$, is defined by

$$\oint_C F\cdot\widehat{n}\; ds = \int_C P(x,y)\; dy - Q(x,y)\; dx,$$

where $\widehat{n}$ denotes the outward unit normal to the curve C, wherever it is defined.

- **The fundamental theorem of Calculus in $\mathbb{R}^2$ or $\mathbb{R}^3$ for oriented triangles**. Let U denote an open region in $\mathbb{R}^2$ or $\mathbb{R}^3$ and T an oriented triangle contained in U. Denote the boundary of T by ∂T. If ω is any differential 1–form defined in U, the

$$\int_T d\omega = \oint_{\partial T} \omega. \tag{1}$$

- **Green's theorem**. Let U denote an open region in $\mathbb{R}^2$ and R be a bounded, open set in U with piece–wise C^1 boundary ∂R contained in U. Assume that ∂R is a simple, closed curve that is oriented in the counterclockwise sense. For any C^1 functions, $P\colon U\to\mathbb{R}$ and $Q\colon U\to\mathbb{R}$, defined in U,

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\, dxdy = \oint_{\partial R} P\; dx + Q\; dy \tag{2}$$

Do the following problems.

1. Let U denote an open subset of $\mathbb{R}^n$ that is path connected; see definition of "path connected" in problem 4 of Assignment #12. Let $F\colon U \to \mathbb{R}^n$ be a vector field with the property that
$$\oint_C F\cdot\, d\overrightarrow{r} = 0,$$
for any simple, piece–wise C^1, closed curve, C, contained in U.

   Let p and q be points in U. Since U is path connected, there exists a C^1 path, $\sigma\colon [0,1] \to U$, connecting p to q. Assume that σ parametrizes a curve C_1 in U. Prove that if $\gamma\colon [0,1] \to U$ is another C^1 path that connects p to q, and $C_2 = \gamma([0,1])$ is paramatrized by γ, then
$$\int_{C_1} F\cdot\, d\overrightarrow{r} = \int_{C_2} F\cdot\, d\overrightarrow{r}.$$

2. Let U denote an open subset of $\mathbb{R}^n$ and let $F\colon U \to \mathbb{R}^n$ be a vector field with the property that $F(v) = \nabla f(v)$ for all $v \in U$, where $f\colon U \to \mathbb{R}$ is a C^1 scalar field.

   Prove that if C is any C^1, simple, closed curve in U, then
$$\oint_C F\cdot\, d\overrightarrow{r} = 0.$$

3. Let T denote the triangle with vertices $P_o(0,0)$, $P_1(2,0)$ and $P_2(1,1)$, where the boundary, ∂T, of T is oriented in the counterclockwise sense. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field given by
$$F(x,y) = -\frac{y}{2}\,\widehat{i} + \frac{x}{2}\,\widehat{j}.$$
Compute the flux of F across ∂T, $\displaystyle\oint_{\partial T} F\cdot d\mathbf{n}$.

4. Let R denote the triangular region in the xy–plane with vertices $(0,0)$, $(1,0)$ and $(1,1)$. Evaluate the double integral $\displaystyle\iint_R xy^2\; dxdy$.

5. Let T and F be as in Problem 3. Evaluate the flux of F across ∂T, $\displaystyle\oint_{\partial T} F\cdot d\mathbf{n}$, by applying Green's Theorem in (2).