Assignment #5

Due on Friday, October 4, 2019

Read Section 3.1 on *Types of Functions in Euclidean Space* in the class Lecture Notes at `http://pages.pomona.edu/~ajr04747/`.

Read Section 3.2 on *Open Subsets of Euclidean Space* in the class Lecture Notes at `http://pages.pomona.edu/~ajr04747/`.

Read Section 3.3 on *Continuous Functions* in the class Lecture Notes at `http://pages.pomona.edu/~ajr04747/`.

Do the following problems

1. Let U_1 and U_2 denote subsets in $\mathbb{R}^n$.

   (a) Show that if U_1 and U_2 are open subsets of $\mathbb{R}^n$, then their intersection

   $$U_1 \cap U_2 = \{y \in \mathbb{R}^n \mid y \in U_1 \text{ and } y \in U_2\}$$

   is also open.

   (b) Show that the set $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; y = 0 \right\}$ is not an open subset of $\mathbb{R}^2$.

2. In Problem 4 of Assignment #3 you proved that every linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}$ must be of the form

   $$T(v) = w \cdot v, \qquad \text{for every } \ v \in \mathbb{R}^n,$$

   where w is some vector in $\mathbb{R}^n$. Use this fact, together with the Cauchy–Schwarz inequality, to prove that T is continuous at every point in $\mathbb{R}^n$.

3. Let x and y denote real numbers.

   Starting with the self–evident inequality: $(|x| - |y|)^2 \geqslant 0$, derive the inequality

   $$|xy| \leqslant \frac{1}{2}(x^2 + y^2). \tag{1}$$

4. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ & \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$

   Use the inequality in (1) to show that f is continuous at the origin.

5. Let
$$f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}, \quad (x,y) \neq (0,0).$$

   Define $f(0,0)$ so that $f(x,y)$ is continuous at $(0,0)$. Explain the reasoning leading to your answer.