Review Problems for Exam 1

1. Compute the (shortest) distance from the point $P(4, 0, -7)$ in $\mathbb{R}^3$ to the plane given by $4x - y - 3z = 12$.

2. Compute the (shortest) distance from the point $P(4, 0, -7)$ in $\mathbb{R}^3$ to the line given by the parametric equations
$$\begin{cases} x &= -1 + 4t, \\ y &= -7t, \\ z &= 2 - t. \end{cases}$$

3. Compute the area of the triangle whose vertices in $\mathbb{R}^3$ are the points $(1, 1, 0)$, $(2, 0, 1)$ and $(0, 3, 1)$

4. Let v and w be two vectors in $\mathbb{R}^3$, and let λ be a scalar. Show that the area of the parallelogram determined by the vectors v and $w + \lambda v$ is the same as that determined by v and w.

5. Let $\widehat{u}$ denote a unit vector in $\mathbb{R}^n$ and $P_{\widehat{u}}(v)$ denote the orthogonal projection of v along the direction of $\widehat{u}$ for any vector $v \in \mathbb{R}^n$. Use the Cauchy–Schwarz inequality to prove that the map
$$v \mapsto P_{\widehat{u}}(v) \quad \text{for all} \quad v \in \mathbb{R}^n$$
is a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

6. Define $f \colon \mathbb{R}^2 \to \mathbb{R}$ by
$$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$
Prove that f is continuous at $(0, 0)$.

7. Show that
$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$
is not continuous at $(0,0)$.

8. Determine the value of L that would make the function
$$f(x,y) = \begin{cases} x\ \sin\left(\dfrac{1}{y}\right) & \text{if } \ y\neq 0; \\ \\ L & \text{otherwise}\ , \end{cases}$$
continuous at $(0,0)$. Is $f\colon \mathbb{R}^2 \to \mathbb{R}$ continuous on $\mathbb{R}^2$? Justify your answer.

9. Define the scalar field $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(v) = \dfrac{1}{2}\|v\|^2$ for all $v \in \mathbb{R}^n$. Show that f is continuous on $\mathbb{R}^n$. Explain the reasoning behind your answer.

10. Define the vector field $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ by
$$F(x,y) = \left(xy,\ \frac{x^2+y}{1+x^2+y^2}\right), \quad \text{for } (x,y)\in\mathbb{R}^2.$$
Show that F is continuous on $\mathbb{R}^2$. Explain the reasoning behind your answer.