Solutions to Review Problems for Exam 1

1. Compute the (shortest) distance from the point $P(4,0,-7)$ in $\mathbb{R}^3$ to the plane given by
$$4x - y - 3z = 12.$$

***Solution**:* The point $P_o(3,0,0)$ is in the plane. Let
$$w = \overrightarrow{P_oP} = \begin{pmatrix} 1 \\ 0 \\ -7 \end{pmatrix}$$

The vector $n = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$ is orthogonal to the plane. To find the shortest distance, d, from P to the plane, we compute the norm of the orthogonal projection of w onto n; that is,
$$d = \|\mathrm{P}_{\widehat{n}}(w)\|,$$
where
$$\widehat{n} = \frac{1}{\sqrt{26}} \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix},$$
a unit vector in the direction of n, and
$$\mathrm{P}_{\widehat{n}}(w) = (w \cdot \widehat{n})\widehat{n}.$$
It then follows that
$$d = |w \cdot \widehat{n}|,$$
where $w \cdot \widehat{n} = \dfrac{1}{\sqrt{26}}(4+21) = \dfrac{25}{\sqrt{26}}$. Hence, $d = \dfrac{25\sqrt{26}}{26} \approx 4.9$. □

2. Compute the (shortest) distance from the point $P(4,0,-7)$ in $\mathbb{R}^3$ to the line given by the parametric equations
$$\begin{cases} x &= -1 + 4t, \\ y &= -7t, \\ z &= 2 - t. \end{cases}$$

***Solution**:* The point $P_o(-1,0,2)$ is on the line. The vector
$$v = \begin{pmatrix} 4 \\ -7 \\ -1 \end{pmatrix}$$

gives the direction of the line. Put

$$w = \overrightarrow{P_oP} = \begin{pmatrix} 5 \\ 0 \\ -9 \end{pmatrix}.$$

The vectors v and w determine a parallelogram whose area is the norm of v times the shortest distance, d, from P to the line determined by v at P_o. We then have that

$$\text{area}(P(v,w)) = \|v\| d,$$

from which we get that

$$d = \frac{\text{area}(P(v,w))}{\|v\|}.$$

On the other hand,

$$\text{area}(P(v,w)) = \|v \times w\|,$$

where

$$v \times w = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ 4 & -7 & -1 \\ 5 & 0 & -9 \end{vmatrix} = 63\widehat{i} + 31\widehat{j} + 35\widehat{k}.$$

Thus, $\|v \times w\| = \sqrt{(63)^2 + (31)^2 + (35)^2} = \sqrt{6155}$ and therefore

$$d = \frac{\sqrt{6155}}{\sqrt{66}} \approx 9.7.$$

□

3. Compute the area of the triangle whose vertices in $\mathbb{R}^3$ are the points $(1,1,0)$, $(2,0,1)$ and $(0,3,1)$

   ***Solution**:* Label the points $P_o(1,1,0)$, $P_1(2,0,1)$ and $P_2(0,3,1)$ and define the vectors

$$v = \overrightarrow{P_oP_1} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad w = \overrightarrow{P_oP_2} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

   The area of the triangle determined by the points P_o, P_1 and P_2 is then half of the area of the parallelogram determined by the vectors v and w. Thus,

$$\text{area}(\triangle P_oP_1P_2) = \frac{1}{2}\|v \times w\|,$$

where

$$v \times w = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ 1 & -1 & 1 \\ -1 & 2 & 1 \end{vmatrix} = -3\widehat{i} - 2\widehat{j} + \widehat{k}.$$

Consequently, $\text{area}(\triangle P_o P_1 P_2) = \dfrac{1}{2}\sqrt{9+4+1} = \dfrac{\sqrt{14}}{2} \approx 1.87.$ □

4. Let v and w be two vectors in $\mathbb{R}^3$, and let λ be a scalar. Show that the area of the parallelogram determined by the vectors v and $w + \lambda v$ is the same as that determined by v and w.

***Solution**:* The area of the parallelogram determined by v and $w + \lambda v$ is

$$\text{area}(P(v, w + \lambda v)) = \|v \times (w + \lambda v)\|,$$

where

$$v \times (w + \lambda v) = v \times w + \lambda v \times v = v \times w.$$

Consequently, $\text{area}(P(v, w + \lambda v)) = \|v \times w\| = \text{area}(P(v, w)).$ □

5. Let $\widehat{u}$ denote a unit vector in $\mathbb{R}^n$ and $P_{\widehat{u}}(v)$ denote the orthogonal projection of v along the direction of $\widehat{u}$ for any vector $v \in \mathbb{R}^n$. Use the Cauchy–Schwarz inequality to prove that the map

$$v \mapsto P_{\widehat{u}}(v) \quad \text{for all} \quad v \in \mathbb{R}^n$$

is a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

***Solution**:* $P_{\widehat{u}}(v) = (v \cdot \widehat{u})\widehat{u}$ for all $v \in \mathbb{R}^n$. Consequently, for any $w, v \in \mathbb{R}^n$,

$$\begin{aligned} P_{\widehat{u}}(w) - P_{\widehat{u}}(v) &= (w \cdot \widehat{u})\widehat{u} - (v \cdot \widehat{u})\widehat{u} \\ &= (w \cdot \widehat{u} - v \cdot \widehat{u})\widehat{u} \\ &= [(w - v) \cdot \widehat{u}]\widehat{u}. \end{aligned}$$

It then follows that

$$\|P_{\widehat{u}}(w) - P_{\widehat{u}}(v)\| = |(w - v) \cdot \widehat{u}|,$$

since $\|\widehat{u}\| = 1$. Hence, by the Cauchy–Schwarz inequality,

$$\|P_{\widehat{u}}(w) - P_{\widehat{u}}(v)\| \leqslant \|w - v\|.$$

Applying the Squeeze Theorem we then get that

$$\lim_{\|w-v\| \to 0} \|P_{\widehat{u}}(w) - P_{\widehat{u}}(v)\| = 0,$$

which shows that $P_{\widehat{u}}$ is continuous at every $v \in V$. □

6. Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that f is continuous at $(0,0)$.

***Solution**:* For $(x,y) \neq (0,0)$

$$\begin{aligned} |f(x,y)| &= \frac{x^2|y|}{x^2+y^2} \\ &\leqslant |y|, \end{aligned}$$

since $x^2 \leqslant x^2 + y^2$ for all $(x,y) \in \mathbb{R}^2$. We then have that, for $(x,y) \neq (0,0)$,

$$|f(x,y)| \leqslant \sqrt{x^2+y^2},$$

which implies that

$$0 \leqslant |f(x,y) - f(0,0)| \leqslant \|(x,y) - (0,0)\|,$$

for $(x,y) \neq (0,0)$. Thus, by the Squeeze Theorem,

$$\lim_{\|(x,y)-(0,0)\| \to 0} |f(x,y) - f(0,0)| = 0,$$

which shows that f is continuous at $(0,0)$. □

7. Show that

$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is not continuous at $(0,0)$.

***Solution**:* Let $\sigma_1(t) = (t,t)$ for all $t \in \mathbb{R}$ and observe that

$$\lim_{t \to 0} \sigma_1(t) = (0,0)$$

and

$$f(\sigma(t)) = 0, \quad \text{for all } t \neq 0.$$

It then follows that

$$\lim_{t\to 0} f(\sigma_1(t)) = 0.$$

Thus, if f were continuous at $(0,0)$, we would have that

$$f(0,0) = 0. \tag{1}$$

On the other hand, if we let $\sigma_2(t) = (t,0)$, we would have that

$$\lim_{t\to 0} \sigma_2(t) = (0,0)$$

and

$$f(\sigma(t)) = 1, \qquad \text{for all } t \neq 0.$$

Thus, if f were continuous at $(0,0)$, we would have that

$$f(0,0) = 1,$$

which is in contradiction with (1). This contradiction shows that f is not continuous at $(0,0)$. □

8. Determine the value of L that would make the function

$$f(x,y) = \begin{cases} x \sin\left(\dfrac{1}{y}\right) & \text{if } y \neq 0; \\ \\ L & \text{otherwise}, \end{cases}$$

continuous at $(0,0)$. Is $f\colon \mathbb{R}^2 \to \mathbb{R}$ continuous on $\mathbb{R}^2$? Justify your answer.

***Solution**:* Observe that, for $y \neq 0$,

$$\begin{aligned} |f(x,y)| &= \left| x \sin\left(\frac{1}{y}\right) \right| \\ &= |x| \left| \sin\left(\frac{1}{y}\right) \right| \\ &\leqslant |x| \\ &\leqslant \sqrt{x^2+y^2}. \end{aligned}$$

It then follows that, for $y \neq 0$,

$$0 \leqslant |f(x,y)| \leqslant \|(x,y)\|.$$

Consequently, by the Squeeze Theorem,

$$\lim_{\|(x,y)\|\to 0} |f(x,y)| = 0.$$

This suggests that we define $L = 0$. If this is the case,

$$\lim_{\|(x,y)\|\to 0} |f(x,y) - f(0,0)| = 0,$$

which shows that f is continuous at $(0,0)$ if $L = 0$.

Next, assume now that $L = 0$ in the definition of f. Then, for any $a \neq 0$, f fails for be continuous at $(a,0)$. To see why this is case, note that for any $y \neq 0$

$$f(a,y) = a\ \sin\left(\frac{1}{y}\right)$$

and the limit of $\sin\left(\frac{1}{y}\right)$ as $y \to 0$ does not exist. □

9. Define the scalar field $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(v) = \frac{1}{2}\|v\|^2$ for all $v \in \mathbb{R}^n$. Show that f is continuous on $\mathbb{R}^n$. Explain the reasoning behind your answer.

   Solution: Define $g\colon \mathbb{R}^n \to \mathbb{R}$ by $g(v) = \|v\|$ for all $v \in \mathbb{R}^n$. This function is continuous everywhere. Indeed, let $u \in \mathbb{R}^n$ and apply the triangle inequality to get the estimate

$$|g(v) - g(u)| \leqslant \|v - u\|, \quad \text{for all } v \in \mathbb{R}^n. \tag{2}$$

   It follows from (2) and the squeeze theorem that

$$\lim_{\|v-u\|\to 0} |g(v) - g(u)| = 0,$$

   which shows that g is continuous at $u \in \mathbb{R}^n$.

   Define $h\colon \mathbb{R} \to \mathbb{R}$ by $h(t) = \frac{1}{2}t^2$, for all $t \in \mathbb{R}$. Then, h is continuous on $\mathbb{R}$ because it is a polynomial function.

   Observe that $f(v) = h(g(v))$, for all $v \in \mathbb{R}^n$; so that, $f = h \circ g$. Thus, f is a composition of continuous functions. Hence, f is continuous. □

10. Define the vector field $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x,y) = \left(xy,\ \frac{x^2+y}{1+x^2+y^2}\right), \quad \text{for } (x,y)\in\mathbb{R}^2.$$

Show that F is continuous on $\mathbb{R}^2$. Explain the reasoning behind your answer.

***Solution**:* Observe that

$$F(x,y) = (f(x,y), g(x,y)), \quad \text{for all } (x,y)\in\mathbb{R}^2, \tag{3}$$

where $f\colon \mathbb{R}^2 \to \mathbb{R}$ is given by

$$f(x,y) = xy, \quad \text{for all } (x,y)\in\mathbb{R}^2, \tag{4}$$

and $g\colon \mathbb{R}^2 \to \mathbb{R}$ is given by

$$g(x,y) = \frac{x^2+y}{1+x^2+y^2}, \quad \text{for all } (x,y)\in\mathbb{R}^2. \tag{5}$$

The function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined in (4) is continuous on $\mathbb{R}^2$ because it is a polynomial function.

The function $g\colon \mathbb{R}^2 \to \mathbb{R}$ is a ratio of polynomial functions for which the denominator in not 0 in $\mathbb{R}^2$; consequently, it is continuous in $\mathbb{R}^2$.

Hence, the components of the vector valued function $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ given in (3) are continuous in $\mathbb{R}^2$; therefore, F is continuous in $\mathbb{R}^2$. □