## Review Problems for Exam 2

1. Define the scalar field $f\colon \mathbb{R}^n \to \mathbb{R}$ by $f(v) = \dfrac{1}{2}\|v\|^2$ for all $v \in \mathbb{R}^n$. Show that $f$ is differentiable on $\mathbb{R}^n$ and compute the linear map $Df(u)\colon \mathbb{R}^n \to \mathbb{R}$ for all $u \in \mathbb{R}^n$. What is the gradient of $f$ at $u$ for all $x \in \mathbb{R}^n$?

2. Let $g\colon\ [0,\infty) \to \mathbb{R}$ be a differentiable, real–valued function of a single variable, and let $f(x,y) = g(r)$ where $r = \sqrt{x^2+y^2}$.

   (a) Compute $\dfrac{\partial r}{\partial x}$ in terms of $x$ and $r$, and $\dfrac{\partial r}{\partial y}$ in terms of $y$ and $r$.

   (b) Compute $\nabla f$ in terms of $g'(r)$, $r$ and the vector $\vec{\mathbf{r}} = x\widehat{\imath} + y\widehat{\jmath}$.

3. Let $f\colon U \to \mathbb{R}$ denote a scalar field defined on an open subset $U$ of $\mathbb{R}^n$, and let $\widehat{u}$ be a unit vector in $\mathbb{R}^n$. If the limit

$$\lim_{t\to 0} \frac{f(v+t\widehat{u}) - f(v)}{t}$$

   exists, we call it the *directional derivative of $f$ at $v$ in the direction of the unit vector $\widehat{u}$.* We denote it by $D_{\widehat{u}}f(v)$.

   (a) Show that if $f$ is differentiable at $v \in U$, then, for any unit vector $\widehat{u}$ in $\mathbb{R}^n$, the directional derivative of $f$ in the direction of $\widehat{u}$ at $v$ exists, and

$$D_{\widehat{u}}f(v) = \nabla f(v) \cdot \widehat{u},$$

   where $\nabla f(v)$ is the gradient of $f$ at $v$.

   (b) Suppose that $f\colon U \to \mathbb{R}$ is differentiable at $v \in U$. Prove that if $D_{\widehat{u}}f(v) = 0$ for every unit vector $\widehat{u}$ in $\mathbb{R}^n$, then $\nabla f(v)$ must be the zero vector.

   (c) Suppose that $f\colon U \to \mathbb{R}$ is differentiable at $v \in U$. Use the Cauchy–Schwarz inequality to show that the largest value of $D_{\widehat{u}}f(v)$ is $\|\nabla f(v)\|$ and it occurs when $\widehat{u}$ is in the direction of $\nabla f(v)$.

4. Let $U$ denote an open and convex subset of $\mathbb{R}^n$. Suppose that $f\colon U \to \mathbb{R}$ is differentiable at every $v \in U$. Fix $u$ and $v$ in $U$, and define $g\colon [0,1] \to \mathbb{R}$ by

$$g(t) = f(u + t(v-u)) \quad \text{for} \quad 0 \leqslant t \leqslant 1.$$

   (a) Explain why the function $g$ is well defined.

(b) Show that $g$ is differentiable on $(0,1)$ and that

$$g'(t) = \nabla f(u + t(v-u)) \cdot (v-u) \quad \text{for} \quad 0 < t < 1.$$

(c) Use the mean value theorem for derivatives to show that there exists a point $z$ is the line segment connecting $u$ to $v$ such that

$$f(v) - f(u) = D_{\widehat{w}} f(z) \|v-u\|,$$

where $\widehat{w}$ is the unit vector in the direction of the vector $v-u$; that is, $\widehat{w} = \dfrac{1}{\|v-u\|}(v-u)$, provided that $v \neq 0$.

(d) Prove that if $U$ is an open and convex subset of $\mathbb{R}^n$, and $f\colon U \to \mathbb{R}$ is differentiable on $U$ with $\nabla f(v) = \mathbf{0}$ for all $v \in U$, then $f$ must be a constant function.

5. Let $U$ be an open subset of $\mathbb{R}^n$ and $I$ be an open interval. Suppose that $f\colon U \to \mathbb{R}$ is a differentiable scalar field and $\sigma\colon I \to \mathbb{R}^n$ be a differentiable path whose image lies in $U$. Suppose also that $\sigma'(t)$ is never the zero vector. Show that if $f$ has a local maximum or a local minimum at some point on the path, then $\nabla f$ is perpendicular to the path at that point.

   *Suggestion:* Consider the real valued function of a single variable $g(t) = f(\sigma(t))$ for all $t \in I$.

6. Let $C$ denote the boundary of the oriented triangle, $T = [(0,0)(1,0)(1,2)]$, in $\mathbb{R}^2$. Evaluate the line integral $\displaystyle\int_C \frac{x^2}{2}\ \mathrm{d}y - \frac{y^2}{2}\ \mathrm{d}x$, by applying the fundamental theorem of Calculus.

7. Let $F(x,y) = 2x\ \widehat{i} - y\ \widehat{j}$ and $R$ be the square in the $xy$–plane with vertices $(0,0)$, $(2,-1)$, $(3,1)$ and $(1,2)$. Evaluate $\displaystyle\oint_{\partial R} F \cdot \widehat{n}\ \ \mathrm{d}s$.

8. Evaluate the line integral $\displaystyle\int_{\partial R} (x^4 + y)\ \ \mathrm{d}x + (2x - y^4)\ \ \mathrm{d}y$, where $R$ is the rectangular region

$$R = \{(x,y) \in \mathbb{R}^2 \mid -1 \leqslant x \leqslant 3,\ -2 \leqslant y \leqslant 1\},$$

   and $\partial R$ is traversed in the counterclockwise sense.

9. Integrate the function given by $f(x,y) = xy^2$ over the region, $R$, defined by:

$$R = \{(x,y) \in \mathbb{R}^2 \mid x \geqslant 0, 0 \leqslant y \leqslant 4 - x^2\}.$$

10. Let $R$ denote the region in the plane defined by inside of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{1}$$

for $a > 0$ and $b > 0$.

(a) Evaluate the line integral $\oint_{\partial R} x \; \mathrm{d}y - y \; \mathrm{d}x$, where $\partial R$ is the ellipse in (1) traversed in the positive sense.

(b) Use your result from part (a) and the fundamental theorem of Calculus to come up with a formula for computing the area of the region enclosed by the ellipse in (1).

11. Evaluate the double integral $\int_R e^{-x^2} \; \mathrm{d}x \; \mathrm{d}y$, where $R$ is the region in the $xy$–plane sketched in Figure 1.

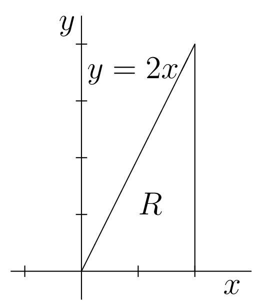


Figure 1: Sketch of Region $R$ in Problem 11